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FINITE ELEMENT APPROXIMATION OF THE TRANSPORT OF REACTIVE SOLUTES IN POROUS MEDIA. PART 1: ERROR ESTIMATES FOR NONEQUILIBRIUM ADSORPTION PROCESSES*

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Abstract. In this paper we analyze a fully practical piecewise linear finite element approximation involving numerical integration, backward Euler time discretization, and possibly regularization of the following degenerate parabolic system arising in a model of reactive solute transport in porous media: find $\{u(x,t), v(x,t)\}$ such that

$$\partial_t u + \partial_t v - \Delta u = f \quad \text{in } \Omega \times (0,T] \qquad u = 0 \quad \text{on } \partial\Omega \times (0,T]$$
$$\partial_t v = k(\varphi(u) - v) \quad \text{in } \Omega \times (0,T]$$
$$u(\cdot\,,0) = g_1(\cdot) \quad v(\cdot\,,0) = g_2(\cdot) \quad \text{in } \Omega \subset \boldsymbol{R}^d, \quad 1 \le d \le 3$$

for given data $k \in \boldsymbol{R}^+$, f, g_1, g_2 and a monotonically increasing $\varphi \in C^0(\boldsymbol{R}) \cap C^1(-\infty,0] \cup (0,\infty)$ satisfying $\varphi(0) = 0$, which is only locally Hölder continuous with exponent $p \in (0,1)$ at the origin, e.g., $\varphi(s) \equiv [s]_+^p$. This lack of Lipschitz continuity at the origin limits the regularity of the unique solution $\{u, v\}$ and leads to difficulties in the finite element error analysis.

Key words. finite elements, error analysis, degenerate parabolic systems, porous medium

AMS subject classifications. 65M15, 65M60, 35K55, 35K65, 35R35

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1. Introduction.

In these papers we study finite element approximations of degenerate parabolic systems and equations as they arise in the modeling of reactive solute transport in porous media as soils or aquifers. The reaction that we are going to take into account is adsorption, that is, a retention/release reaction of the solute, e.g., a contaminant, with the porous skeleton. Adsorption is a major concern in soil science and hydrology as it is often the primary factor determining the mobility of a solute.

We consider the process on a macroscopic level, i.e., averaged/homogenized scale, where single grains and pores do not appear anymore. A macroscopic model has the form (cf. [10] and [7] for a derivation)

$$\partial_t(\Theta u) + \rho \partial_t[\lambda_1 \psi(u) + \lambda_2 v] - \nabla.(\Theta \underline{\underline{D}} \nabla u - \underline{q} u) = f \quad \text{in } Q_T, \tag{1.1a}$$
$$\partial_t v = r(u,v) \quad \text{in } Q_T, \tag{1.1b}$$

supplemented by initial conditions for u and v and appropriate boundary conditions for u. Here u and v are the unknowns of the system, the dissolved concentration (with reference to the water-filled part of the pore space) and the adsorbed concentration in nonequilibrium (with reference to the mass of the porous skeleton). The process takes place in a bounded domain Ω in $\boldsymbol{R}^d$, $1 \le d \le 3$. Let $[0,T]$ be the fixed time interval and $Q_t \equiv \Omega \times (0,t]$ for $t \in (0,T]$. The other quantities, all assumed to be

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known, describe either the underlying water flow regime and geology, as the water content Θ, the volumetric water flux $\underline{q}$, the sum of diffusion and dispersion matrix $\underline{\underline{D}}$, and the bulk density ρ, or the adsorption process: here it is assumed that two classes of adsorption sites may be distinguished (with relative specific grain surfaces $\lambda_i \in [0,1]$). The sites in class 2 are in (chemical) nonequilibrium, and the kinetics are described by (1.1b), which applies to adsorption reaction at a time scale comparable with transport. For sites where the reaction is considerably faster, a quasi-stationary approach is feasible, assuming the reaction to be equilibrium. This approach is used for sites in class 1, leading to an algebraic expression for the adsorbed concentration in terms of the dissolved concentration—the adsorption isotherm ψ.

A common heuristic approach for the rate function r consists of taking it proportional to the deviation from equilibrium, i.e.,

$$r(u,v) \equiv k(\varphi(u) - v), \tag{1.2}$$

where φ is the adsorption isotherm for sites of class 2 and $k > 0$ is a rate parameter. We will restrict ourselves to this form, noting that the presented analysis exploits this structure. The quasi-linear, respectively semilinear (for $\lambda_1 = 0$), system (1.1) may be degenerate because there are typical examples for the isotherms φ or ψ, which are not Lipschitz continuous at $u = 0$ such as is the Freundlich isotherm

$$\varphi(u) \equiv \alpha u^p \quad \text{for } u \geq 0, \text{ where } \alpha \in \boldsymbol{R}^+ \text{ and } p \in (0,1). \tag{1.3}$$

On the other hand isotherms are monotone increasing, such that in the following we will consider monotone nonlinearities allowing for degenerate behavior like (1.3) at the origin.

In the first part of this paper we consider only nonequilibrium adsorption, such that we assume $\lambda_1 = 0$ from now on. The underlying water flow regime in general leads to time- and space-dependent coefficients, but with a linear uniformly parabolic operator on u due to

$$\partial_t \Theta + \nabla.\underline{q} = 0, \qquad \Theta(x,t) \geq \Theta_0 > 0 \quad \text{in } Q_T. \tag{1.4}$$

The degenerate semilinear system (1.1), supplemented by inflow and outflow conditions has been extensively studied by [10] and [5]. The boundary conditions read as

$$(\Theta \underline{\underline{D}} \nabla u - \underline{q} u).\underline{n} = F \text{ on } S_1 \times (0,T] \quad \text{and} \quad \underline{\underline{D}} \nabla u.\underline{n} = 0 \text{ on } S_2 \times (0,T], \tag{1.5}$$

where $\underline{n}$ is the outward normal to $\partial\Omega \equiv S_1 \cup S_2$ and S_1 is defined by $\underline{q}.\underline{n} \leq 0$ (the inflow boundary) and S_2 by $\underline{q}.\underline{n} \geq 0$ (the outflow/noflow boundary). A specific sequence of testing leads to a uniqueness result (see II Thm. 2.2 in [10]), which can be extended to the usual energy norm stability estimate for the u-components of the solutions but only under certain structural conditions on the coefficients (II Thm. 2.6 in [10]). These conditions are fulfilled for time-independent coefficients, i.e., for stationary water flow.

Our aim is to prove order of convergence estimates in energy norms for the corresponding finite element approximation; therefore, we consider this stability estimate to be important. In fact it turns out that the same approach enables us to reduce the error estimation (for the continuous in time conforming Galerkin approximation) to problems which have already been studied in [3]; see [11] for a preliminary account. In fact the problem considered in [3] can be viewed as a stationary version of the present problem by neglecting the desorption term $-kv$. Therefore we restrict ourselves to situations where this reasoning for the stability estimate is possible by considering only stationary water flow. We substantially extend and refine the aforementioned preliminary analysis by improving on the error bounds there and considering a fully practical scheme involving numerical integration on the nonlinear term and time discretization using the backward Euler method. The analysis is centered on introducing a regularized system (P_ε) obtained by substituting φ by a Lipschitz continuous φ_ε, differing only near $u = 0$. In fact if the solution u satisfies a nondegeneracy (N.D.) condition (see below), by adapting the regularization parameter ε to the discretization parameters one can prove better rates of convergence for the approximation of (P_ε) to (P) than for the approximation of (P) directly. This situation is not uncommon for the finite element approximation of degenerate problems (e.g., see [14]).

The non-Lipschitzian behavior of φ at $u = 0$ can only play an important role if fronts, given by the boundary of the support of u (or v) in Ω, do not vanish instantaneously, as for the heat equation, but are preserved, i.e., if the problem exhibits a finite speed of propagation property. This property is analyzed in [10] for the one-dimensional case and found to be characterized by

$$\Phi^{-\frac{1}{2}} \in L^1(0, \delta) \quad \text{for some } \delta > 0, \tag{1.6}$$

where $\Phi(s) \equiv \int_0^s \varphi(\sigma)\, d\sigma$. This is fulfilled by the example (1.3) and may be considered as the typical case in the following. The nondegeneracy condition describes the minimal growth of u away from the front. This local behavior of the profile has only been analyzed for traveling wave solutions (see [6]). We will assume later on that φ is Hölder continuous near $u = 0$ with exponent $p \in (0, 1)$. If in addition the exponent is sharp, i.e.,

$$\varphi(u) \geq \alpha u^p \quad \text{for } u \in [0, \delta_0] \text{ and for some } \alpha, \delta_0 > 0, \tag{1.7}$$

then

$$A_\varepsilon(t) \leq C\varepsilon^{\frac{1}{2}}, \qquad \text{(N.D.)} \tag{1.8a}$$

where

$$A_\varepsilon(t) \equiv \int_0^t \underline{m}(\Omega_\varepsilon(s))\, ds, \tag{1.8b}$$

$$\Omega_\varepsilon(t) \equiv \{x \in \Omega : u(x, t) \in (0, \varepsilon^{1/(1-p)})\}, \tag{1.8c}$$

and $\underline{m}$ is the Lebesgue measure.

Our analysis applies to the case of general time-independent coefficients (assuming they are sufficiently regular). However, the fact that we analyze the Galerkin procedure implies the requirement that the process is not convection dominated where we would encounter the well-known difficulties. There are alternative procedures for this situation like the streamline diffusion method or the modified method of characteristics. We expect that the techniques that we are going to develop here will enable us to analyze also variants of these methods. We refer to [4] for a first account with respect to the modified method of characteristics.

For ease of exposition we will develop our results for the following model problem, which keeps the specific difficulty of the non-Lipschitz nonlinearity in the form (1.2) but reduces the handling of standard terms.

(P) Find $\{u(x, t), v(x, t)\}$ such that

$$\begin{gathered} \partial_t u + \partial_t v - \Delta u = f \quad \text{in } Q_T \qquad u = 0 \quad \text{on } \partial\Omega \times (0, T] \\ \partial_t v = k(\varphi(u) - v) \quad \text{in } Q_T \\ u(\cdot\,, 0) = g_1(\cdot) \quad v(\cdot\,, 0) = g_2(\cdot) \quad \text{in } \Omega, \end{gathered}$$

where we make the following assumptions on the given data.

(D1): $\Omega \subset \boldsymbol{R}^d$, $1 \leq d \leq 3$, with either Ω convex polyhedral or $\partial\Omega \in C^{1,1}$, $k \in \boldsymbol{R}^+$, $f \in L^\infty(Q_T)$, $g_1 \in L^\infty(\Omega) \cap H_0^1(\Omega)$, $g_2 \in L^\infty(\Omega)$, and $\varphi \in C^0(\boldsymbol{R})$, is such that

$$\varphi(0) = 0,\ \varphi(s) > 0 \quad \forall s > 0 \text{ and } \varphi \text{ is monotonically increasing,} \tag{1.9a}$$

$$\varphi \in C^1(-\infty, 0] \cup (0, \infty), \tag{1.9b}$$

$$\begin{array}{l}\text{there exist } L \in \boldsymbol{R}^+ \text{ and } \varepsilon_0,\ p \in (0,1] \text{ such that} \\ |\varphi(a) - \varphi(b)| \leq L|a-b|^p \quad \text{for all } a, b \in [0, \varepsilon_0].\end{array} \tag{1.9c}$$

Below we gather together the further assumptions that we require at various stages in the paper.

(D2): In addition to (D1) we assume that $f \in H^1(0,T;L^2(\Omega))$ and $g_1 \in H^2(\Omega)$ and to simplify the analysis that $k \geq k_0$.

(D3): In addition to (D2) we assume that the constant M in (2.1c) can be chosen uniformly for all $s \in \boldsymbol{R}$. (In view of the bounds (2.5a) for u, see Theorem 2.2; this is always achievable by changing $\varphi(s)$ for $|s| \geq m = \max\{-\underline{u}, \overline{u}\}$.) Let Ω^h be a polyhedral approximation to Ω defined by $\bar{\Omega}^h \equiv \bigcup_{\kappa \in T^h} \bar{\kappa}$ with $\mathrm{dist}(\partial\Omega, \partial\Omega^h) \leq Ch^2$, where T^h is a partitioning consisting of regular simplices κ with $h_\kappa \equiv \mathrm{diam}(\kappa)$ and $h \equiv \max_{\kappa \in T^h} h_\kappa$. For ease of exposition we assume that $\Omega^h \subseteq \Omega$.

(D4): In addition to (D3) we assume that $f \in H^1(0,T;C^0(\bar{\Omega})) \cap L^2(0,T;H^2(\Omega))$ and $g_2 \in C^0(\bar{\Omega})$.

Then we prove our basic error bound for a fully practical approximation to (P) under assumptions (D5) and $g_2 \in H^2(\Omega)$, where we have the following.

(D5): In addition to (D4) we assume that T^h is such that

(i) for $d = 2$ it is weakly acute; that is, for any pair of adjacent triangles the sum of opposite angles relative to the common side does not exceed π,

(ii) for $d = 3$ the angle between any two faces of the same tetrahedron does not exceed $\pi/2$.

We improve on these basic error bounds by replacing (D5) by (D6).

(D6): In addition to (D5) we assume that

(i) $\Omega \subset \boldsymbol{R}^d$, $d = 1$ or 2, and T^h is a quasi-uniform partition if $d = 2$,

(ii) g_1, g_2, and $f \geq 0$,

(ii) $\varphi \in C^2(0, m]$ such that $\varphi''(s) \leq 0$ for all $s \in (0, m]$, where $m = \max\{\overline{u}\}$; see (2.5a).

By the last assumption in (D2) we do not neglect any important features, as for $k \to 0$ we expect convergence to the case of no reaction, i.e., to the linear diffusion equation. We note that in proving the error bounds in this paper, as opposed to bounds (3.13) and (4.1), the only place that the acuteness assumption on the partitioning T^h is required is in establishing the bounds (3.11) and (3.12).

The layout of this paper is as follows. In the next section we establish the existence and uniqueness of a solution to (P) under assumptions (D1) by first establishing these results for a regularized version (P_ε). In addition we establish a number of useful a priori estimates for (P_ε) under assumptions (D1) and (D2). In section 3 under assumptions (D5) and $g_2 \in H^2(\Omega)$ we prove error bounds for a continuous in time continuous piecewise linear finite element approximation in space, involving numerical integration, of (P_ε). Moreover, we improve on these bounds if (D6) holds in place of (D5). In addition we note that one can prove superior error bounds for a less practical scheme involving no numerical integration under the weaker assumptions: $g_2 \in H^1(\Omega)$ and (D3) holds. In section 4 we consider a fully practical approximation involving discretization in time using the backward Euler method. Finally, in section 5 we report on a numerical experiment. We note that in the interest of brevity some proofs are omitted or just sketched. However, full proofs can be found in the self-contained report [2].

Throughout the paper we adopt the standard notation for Sobolev spaces. We note that the seminorm $|\cdot|_{H^1(\Omega)}$ and norm $\|\cdot\|_{H^1(\Omega)}$ are equivalent on $H^1_0(\Omega)$. The standard L^2 inner product over Ω is denoted by $\langle\cdot,\cdot\rangle$. Throughout C or C_i denote generic positive constants independent of ε the regularization parameter, h the mesh spacing, and k the reaction rate parameter. If a constant does depend on k, say, this will be written as $C(k)$. We track the constant k in the analysis as we use nearly all the results in this paper to study the case of k infinite, equilibrium adsorption, in part 2 of this paper [18]. This often makes the present analysis more complicated than it need be if we were just interested in the case k finite. Similarly, we isolate the assumption on the data g_2 for the purposes of part 2.

The infinite k case is closely related to a degenerate parabolic problem which has been investigated intensively, the (generalized) porous medium equation. The most complete order of convergence analysis until now for the finite element approximation of this equation, taking into account time discretization and numerical integration, is to be found in [14]. In fact many of the proof techniques, e.g., regularization, used in this paper are similar to those used there. In addition we note similarities with [16], where a fully practical approximation of a phase-relaxed Stefan problem is analyzed, that is, essentially problem (P) with the second equation replaced by

$$\partial_t v + H^{-1}(v) \ni r(u,v) \quad \text{in } Q_T, \tag{1.10}$$

where $H(\cdot)$ is the Heaviside graph and $r(\cdot\,,\cdot)$ is locally Lipschitz with $r(0,v)=0$ and $r(\cdot\,,v)$ strictly increasing in a neighborhood of 0 for all $v\in[0,1]$, v being the phase variable. The key novelty in the present problem is the special non-Lipschitz nature of the nonequilibrium adsorption rate.

2. The continuous problem. In this section we establish the existence and uniqueness of a solution to (P) and a number of useful a priori bounds. First we introduce a regularized version of (P) for $\varepsilon\in[0,\varepsilon_0]$ (ε_0 as in (1.9c)).

($\mathbf{P}_\varepsilon$) Find $\{u_\varepsilon(x,t), v_\varepsilon(x,t)\}$ such that

$$\begin{gathered}\partial_t u_\varepsilon + \partial_t v_\varepsilon - \Delta u_\varepsilon = f \quad \text{in } Q_T \qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega\times(0,T],\\ \partial_t v_\varepsilon = k(\varphi_\varepsilon(u_\varepsilon)-v_\varepsilon) \quad \text{in } Q_T,\\ u_\varepsilon(\cdot\,,0)=g_1(\cdot) \quad v_\varepsilon(\cdot\,,0)=g_2(\cdot) \quad \text{in } \Omega,\end{gathered}$$

where $\varphi_\varepsilon\in C^{0,1}_{\text{loc}}(\boldsymbol{R})$ is such that

$$\varphi_\varepsilon(s)\equiv\varphi(s) \quad \text{for } s\notin(0,\varepsilon^{1/(1-p)}), \tag{2.1a}$$

$$\varphi_\varepsilon(s) \text{ is strictly monotonically increasing on } [0,\varepsilon^{1/(1-p)}], \tag{2.1b}$$

$$\begin{gathered}\text{for } m\in\boldsymbol{N} \text{ there exists an } M(m)\in\boldsymbol{R}^+:\\ \varphi_\varepsilon(b)-\varphi_\varepsilon(a)\le M(m)\varepsilon^{-1}(b-a) \qquad \text{for } -m\le a\le b\le m.\end{gathered} \tag{2.1c}$$

Note that M can be chosen independently of m if φ' is bounded in $\boldsymbol{R}\setminus(0,\delta)$ for some $\delta>0$. In addition we set

$$\Phi_\varepsilon(s)\equiv\int_0^s\varphi_\varepsilon(\sigma)\,d\sigma. \tag{2.2}$$

It is a simple matter to deduce from the conditions (2.1) that for all $|a|,|b|\le m$,

$$[M(m)]^{-1}\varepsilon|\varphi_\varepsilon(a)-\varphi_\varepsilon(b)|^2\le[\varphi_\varepsilon(a)-\varphi_\varepsilon(b)](a-b)\le M(m)\varepsilon^{-1}|a-b|^2 \tag{2.3a}$$

and

$$\varphi_\varepsilon(\varepsilon^{1/(1-p)}) \equiv \varphi(\varepsilon^{1/(1-p)}) \leq L\varepsilon^{p/(1-p)}, \tag{2.3b}$$

with L as in (1.9c). The simplest choice for φ_ε is the linear regularization

$$\varphi_\varepsilon(s) \equiv \varepsilon^{-1/(1-p)}\varphi(\varepsilon^{1/(1-p)})s \quad \text{for } s \in (0, \varepsilon^{1/(1-p)}). \tag{2.4}$$

For $k \in \boldsymbol{R}^+$ and for sufficiently smooth w we set

$$\|w\|^2_{E_1(k,t)} \equiv |w|^2_{L^2(Q_T)} + \frac{1}{2}k^{-1}|w(\cdot\,,t)|^2_{L^2(\Omega)}$$

and

$$\|w\|^2_{E_2(k,t)} \equiv \|w\|^2_{E_1(k,t)} + \frac{1}{2}\left|\nabla \int_0^t w(\cdot\,,s)\,ds\right|^2_{L^2(\Omega)} + k^{-1}|\nabla w|^2_{L^2(Q_t)}.$$

DEFINITION 2.1. *$\{u_\varepsilon, v_\varepsilon\}$ is a weak upper (lower) solution to* (P_ε) *if $u_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \equiv W_2^{1,1}(Q_T)$, $\varphi_\varepsilon(u_\varepsilon) \in L^2(Q_T)$ and $v_\varepsilon \in H^1(0,T;L^2(\Omega))$ are such that for all test functions $\eta \in L^2(0,T;H_0^1(\Omega))$ with $\eta \geq 0$ in Q_T,*

$$\int_{Q_T} [\partial_t u_\varepsilon \eta + \nabla u_\varepsilon . \nabla \eta + k(\varphi_\varepsilon(u_\varepsilon) - v_\varepsilon)\eta - f\eta]\,dx\,dt \geq (\leq)\,0,$$
$$u_\varepsilon \geq (\leq)\,0 \quad \text{on } \partial\Omega \times (0,T],$$
$$\partial_t v_\varepsilon \geq (\leq)\,k(\varphi_\varepsilon(u_\varepsilon) - v_\varepsilon) \quad \text{in } Q_T,$$
$$u_\varepsilon(\cdot\,,0) \geq (\leq)\,g_1(\cdot) \qquad v_\varepsilon(\cdot\,,0) \geq (\leq)\,g_2(\cdot) \quad \text{in } \Omega.$$

$\{u_\varepsilon, v_\varepsilon\}$ is a weak solution to (P_ε) *if it is both a weak lower solution and a weak upper solution to* (P_ε). *Similar definitions hold for* (P) *with φ_ε in the above replaced by φ.*

THEOREM 2.1. *Let the assumptions* (D1) *hold. Then for all $\varepsilon \in (0, \varepsilon_0]$ there exists a unique weak solution $\{u_\varepsilon, v_\varepsilon\}$ to* (P_ε) *such that*

$$\underline{u} \leq u_\varepsilon \leq \overline{u} \quad \text{and} \quad \underline{v} \leq v_\varepsilon \leq \overline{v} \quad \text{in } Q_T, \tag{2.5a}$$

$$|\nabla u_\varepsilon|^2_{L^2(Q_T)} + |\partial_t u_\varepsilon|^2_{L^2(Q_T)} + |\partial_t v_\varepsilon|^2_{L^\infty(Q_T)} \leq C(k), \tag{2.5b}$$

where $\underline{u}$, $\overline{u}$, $\underline{v}$, $\overline{v} \in C^0[0,T]$ are all independent of ε and bounded uniformly in k. Furthermore, if g_1, g_2, and $f \geq 0$ one can take $\underline{u} = \underline{v} = 0$.

Proof. Existence of a solution to (P_ε) with flux boundary conditions, (1.5), can be found in [10]. The proof consists of finding weak lower and upper solutions, $\{\underline{u}, \underline{v}\}$ and $\{\overline{u}, \overline{v}\}$, of ($\mathrm{P}_\varepsilon$) and then applying the Schauder fixed point theorem. A modification of this proof for the present Dirichlet boundary conditions, leading to the bounds (2.5a, 2.5b), can be found in [2].

We now prove uniqueness. Assume there exist two weak solutions $\{u_\varepsilon^{(i)}, v_\varepsilon^{(i)}\}$, $i = 1, 2$ to (P_ε). Let $e_\varepsilon^u \equiv u_\varepsilon^{(1)} - u_\varepsilon^{(2)}$ and $e_\varepsilon^v \equiv v_\varepsilon^{(1)} - v_\varepsilon^{(2)}$. Subtracting the first equations in (P_ε), using the test function $\eta(\cdot\,,s) \equiv k^{-1}e_\varepsilon^u(\cdot\,,s) + \int_s^t e_\varepsilon^u(\cdot\,,\sigma)\,d\sigma$ for $s \in [0,t]$ and $\eta(\cdot\,,s) \equiv 0$ for $s \in (t,T]$ and performing integration by parts yields that

$$\begin{aligned} \|e_\varepsilon^u\|^2_{E_2(k,t)} &= -\int_0^t \left\langle k^{-1}\partial_s e_\varepsilon^v(\cdot\,,s) + e_\varepsilon^v(\cdot\,,s), e_\varepsilon^u(\cdot\,,s)\right\rangle ds \\ &= -\int_0^t \left\langle \varphi_\varepsilon(u_\varepsilon^1(\cdot\,,s)) - \varphi_\varepsilon(u_\varepsilon^2(\cdot\,,s)), e_\varepsilon^u(\cdot\,,s)\right\rangle ds. \end{aligned} \tag{2.6}$$

From (2.6), (2.3a), and (2.5a) it follows that $u_\varepsilon^{(1)} = u_\varepsilon^{(2)}$ and hence $v_\varepsilon^{(1)} = v_\varepsilon^{(2)}$. ∎

We note that if specific L^∞ bounds like (2.5a) are not required, one can deal with problems involving a more general rate function, e.g., see [17], where (1.10) is considered.

Remark 2.1. Bounds $\underline{u}$, $\overline{u}$, $\underline{v}$, $\overline{v} \in C^0(\bar{\Omega})$ independent of ε, k, and T are available if one of the following conditions is satisfied.

For $\{\overline{u}, \overline{v}\}$: either $\varphi(\infty) = \infty$ or $\varphi(g_1) \geq g_2$ or $\varphi_\varepsilon(g_1) \geq g_2$ in Ω.

For $\{\underline{u}, \underline{v}\}$: either $\varphi(-\infty) = -\infty$ or $\varphi(g_1) \leq g_2$ or $\varphi_\varepsilon(g_1) \leq g_2$ in Ω.

This can be seen as follows. Let $w \in H^2(\Omega)$ $(\subset C^0(\bar{\Omega}))$ be such that $-\Delta w = 1$ in Ω and $w = 1$ on $\partial\Omega$. It follows that $w \geq 1$ in Ω. Let $\overline{\gamma} \in \boldsymbol{R}$ be such that $\overline{\gamma} \geq 1$, $\overline{\gamma} \geq g_1$ in Ω, $\overline{\gamma} \geq f$ in Q_T, and if $\varphi(\infty) = \infty$ also $\varphi_\varepsilon(\overline{\gamma}) = \varphi(\overline{\gamma}) \geq g_2$ in Ω. Then $\overline{u} \equiv \overline{\gamma} w$ and $\overline{v} \equiv \varphi(\overline{u}) = \varphi_\varepsilon(\overline{u})$ as $\overline{u} \geq 1$ defines an upper solution of (P_ε). Note that either $\overline{v} \equiv \varphi_\varepsilon(\overline{u}) \geq \varphi_\varepsilon(\overline{\gamma}) \geq g_2$ or $\overline{v} \equiv \varphi(\overline{u}) \geq \varphi(g_1) \geq g_2$ or $\overline{v} \equiv \varphi_\varepsilon(\overline{u}) \geq \varphi_\varepsilon(g_1) \geq g_2$ in Ω depending on which condition is satisfied. Analogously, $\underline{u} \equiv \underline{\gamma} w$ and $\underline{v} \equiv \varphi(\underline{u})$ is a lower solution of (P_ε), where $\underline{\gamma} \in \boldsymbol{R}$ is chosen such that $\underline{\gamma} \leq 0$, $\underline{\gamma} \leq g_1$ in Ω, $\underline{\gamma} \leq f$ in Q_T, and if $\varphi(-\infty) = -\infty$ also $\varphi_\varepsilon(\underline{\gamma}) = \varphi(\underline{\gamma}) \leq g_2$ in Ω. $\square$

THEOREM 2.2. *Let the assumptions* (D1) *hold. Then there exists a unique weak solution* $\{u, v\}$ *to* (P) *and for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in (0, T]$,

$$\|u - u_\varepsilon\|^2_{E_2(k,t)} + \varepsilon|\varphi(u) - \varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} + \varepsilon\|v - v_\varepsilon\|^2_{E_1(k,t)} \leq CA_\varepsilon(t)\varepsilon^{(1+p)/(1-p)}. \tag{2.7}$$

In addition, the bounds (2.5a, 2.5b) *hold true for* $\{u, v\}$ *and in particular if* g_1, g_2, *and* $f \geq 0$ *then* u, $v \geq 0$ *in* Q_T.

Proof. Existence of a solution $\{u, v\}$ follows by letting $\varepsilon \to 0$ in (P_ε), from which it is clearly seen that bounds (2.5a, 2.5b) hold true for $\{u, v\}$; see [2] for details. Uniqueness of a solution to (P) follows as for (P_ε), that is, (2.6) with $\varepsilon = 0$ and noting (1.9a).

The proof of (2.7) is similar to that of uniqueness. Let $e^u \equiv u - u_\varepsilon$ and $e^v \equiv v - v_\varepsilon$. Subtracting the first equation in (P_ε) from that in (P), using the test function $\eta(\cdot, s) \equiv k^{-1} e^u(\cdot, s) + \int_s^t e^u(\cdot, \sigma)\, d\sigma$ for $s \in [0, t]$, $\eta(\cdot, s) \equiv 0$ for $s \in (t, T]$, and performing integration by parts yields that

$$\|e^u\|^2_{E_2(k,t)} = -\int_0^t \langle \varphi(u(\cdot, s)) - \varphi_\varepsilon(u_\varepsilon(\cdot, s)), e^u(\cdot, s)\rangle\, ds. \tag{2.8}$$

Noting that $\varphi(u) \equiv \varphi_\varepsilon(\zeta)$, where $\zeta \equiv \varphi_\varepsilon^{-1}(\varphi(u))$ if $\varphi(u) \in (0, \varphi(\varepsilon^{1/(1-p)}))$ and $\zeta \equiv u$ otherwise, it follows from (2.8) and (2.3) that

$$\begin{aligned} &\|e^u\|^2_{E_2(k,t)} + [M(m)]^{-1}\varepsilon|\varphi(u) - \varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} \\ &\quad \leq \int_0^t \langle \varphi(u(\cdot, s)) - \varphi_\varepsilon(u_\varepsilon(\cdot, s)), (\zeta - u)(\cdot, s)\rangle\, ds \\ &\quad \leq \frac{1}{2}[M(m)]^{-1}\varepsilon|\varphi(u) - \varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} + \frac{1}{2}M(m)\varepsilon^{-1}|\zeta - u|^2_{L^2(Q_t)} \\ &\quad \leq M(m)\varepsilon^{-1}|\zeta - u|^2_{L^2(Q_t)} \leq CA_\varepsilon(t)\varepsilon^{-1}\varepsilon^{2/(1-p)}, \end{aligned} \tag{2.9}$$

where $[\inf \underline{u}, \sup \overline{u}] \subseteq [-m, m]$; see (2.1c) and Theorem 2.1. Finally, subtracting the second equation in (P_ε) from that in (P), multiplying by e^v, and integrating over Q_t yields

$$\|e^v\|^2_{E_1(k,t)} = \int_0^t \langle \varphi(u(\cdot, s)) - \varphi_\varepsilon(u_\varepsilon(\cdot, s)), e^v(\cdot, s)\rangle\, ds \leq C|\varphi(u) - \varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)}. \tag{2.10}$$

Combining (2.9) and (2.10) yields the desired result (2.7). $\square$

Because of the bounds in (2.5a) we now can fix M in (2.1c) when dealing with u or u_ε. We end this section by proving some useful bounds on the unique weak solution $\{u_\varepsilon, v_\varepsilon\}$ of (P_ε), $\varepsilon \in (0, \varepsilon_0]$.

LEMMA 2.1. *Under assumptions* (D1) *we have for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in (0, T]$ *that*

$$\begin{aligned} &\varepsilon|\nabla\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_T)} + \int_0^T \langle \nabla u_\varepsilon(\cdot\,,s), \nabla\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))\rangle\, ds + \langle \Phi_\varepsilon(u_\varepsilon(\cdot\,,t)), 1\rangle \\ &\quad + k|\varphi_\varepsilon(u_\varepsilon) - v_\varepsilon|^2_{L^2(Q_T)} + |v_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} + k^{-1}|\partial_t v_\varepsilon|^2_{L^2(Q_T)} \le C. \end{aligned} \tag{2.11}$$

Proof. From (P_ε) we have that

$$\begin{aligned} &\int_0^t [\langle \nabla u_\varepsilon(\cdot\,,s), \nabla\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))\rangle + \langle \partial_s u_\varepsilon(\cdot\,,s), \varphi_\varepsilon(u_\varepsilon(\cdot\,,s))\rangle \\ &\qquad + \langle \partial_s v_\varepsilon(\cdot\,,s), v_\varepsilon(\cdot\,,s)\rangle + k|\varphi_\varepsilon(u_\varepsilon(\cdot\,,s)) - v_\varepsilon(\cdot\,,s)|^2_{L^2(\Omega)}]\, ds \\ &\quad = \int_0^t \langle f(\cdot\,,s), \varphi_\varepsilon(u_\varepsilon(\cdot\,,s))\rangle\, ds. \end{aligned} \tag{2.12}$$

From (2.1) it follows for all $w \in H^1_0(\Omega)$ with $|w(x)| \le m$ for $x \in \Omega$ that $\varphi_\varepsilon(w) \in H^1_0(\Omega)$; see, e.g., Theorem 7.8 and section 7.5 in [9] and 5.12.5 in [12], and

$$[M(m)]^{-1}\varepsilon|\nabla\varphi_\varepsilon(w)|^2_{L^2(\Omega)} \le \langle \nabla w, \nabla\varphi_\varepsilon(w)\rangle. \tag{2.13}$$

Noting (2.13) and (2.5a) yield that

$$\begin{aligned} &\varepsilon|\nabla\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} + \int_0^t \langle \nabla u_\varepsilon(\cdot\,,s), \nabla\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))\rangle\, ds + \langle \Phi_\varepsilon(u_\varepsilon(\cdot\,,t)), 1\rangle \\ &\qquad + |v_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} + k|\varphi_\varepsilon(u_\varepsilon) - v_\varepsilon|^2_{L^2(Q_t)} \\ &\quad \le C\left[\int_0^t \langle f(\cdot\,,s), \varphi_\varepsilon(u_\varepsilon(\cdot\,,s))\rangle\, ds + \langle \Phi_\varepsilon(u_\varepsilon(\cdot\,,0)), 1\rangle + |v_\varepsilon(\cdot\,,0)|^2_{L^2(\Omega)}\right] \\ &\quad \le C_1 + C_2|\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} \\ &\quad \le C_1 + C_2|\varphi_\varepsilon(u_\varepsilon) - v_\varepsilon|^2_{L^2(Q_t)} + C_2|v_\varepsilon|^2_{L^2(Q_t)} \le C_3, \end{aligned} \tag{2.14}$$

where we can choose C_2 sufficiently small. Hence the desired result (2.11) then follows from (2.14) and the second equation in (P_ε). ∎

LEMMA 2.2. *Under assumptions* (D2) *we have for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $t \in (0, T]$ *that*

$$\begin{aligned} &|\nabla u_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} + |\partial_t u_\varepsilon|^2_{L^2(Q_T)} + \varepsilon|\partial_t v_\varepsilon|^2_{L^2(Q_T)} + \varepsilon|\partial_t[\varphi_\varepsilon(u_\varepsilon)]|^2_{L^2(Q_T)} \\ &\qquad + k^{-1}[|\partial_t u_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} + \varepsilon|\partial_t v_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} + |\nabla(\partial_t u_\varepsilon)|^2_{L^2(Q_T)}] \\ &\quad \le C[1 + k|\varphi_\varepsilon(g_1) - g_2|^2_{L^2(\Omega)}] \le Ck. \end{aligned} \tag{2.15}$$

Proof. Differentiating the first equation in (P_ε) with respect to t yields that

$$k^{-1}\partial_{tt}u_\varepsilon + (\varphi'_\varepsilon(u_\varepsilon)\partial_t u_\varepsilon - \partial_t v_\varepsilon) - k^{-1}\Delta(\partial_t u_\varepsilon) = k^{-1}\partial_t f \quad \text{in } Q_T \tag{2.16a}$$

and hence that

$$k^{-1}\partial_{tt}u_\varepsilon + (1 + \varphi'_\varepsilon(u_\varepsilon))\partial_t u_\varepsilon - \Delta(k^{-1}\partial_t u_\varepsilon + u_\varepsilon) = k^{-1}\partial_t f + f \quad \text{in } Q_T. \tag{2.16b}$$

This formal procedure can be justified as follows. Consider the auxiliary linear initial-boundary value problem: find w_ε such that

$$k^{-1}\partial_t w_\varepsilon - k^{-1}\Delta w_\varepsilon = k^{-1}\partial_t f - \varphi'_\varepsilon(u_\varepsilon)\partial_t u_\varepsilon + \partial_t v_\varepsilon \quad \text{in } Q_T,$$

$$w_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T], \qquad w_\varepsilon(\cdot\,,0) = \Delta g_1(\cdot) + f(\cdot\,,0) - k[\varphi_\varepsilon(g_1(\cdot)) - g_2(\cdot)] \quad \text{in } \Omega.$$

Due to (2.5) and (2.1c), $\partial_t[\varphi_\varepsilon(u_\varepsilon)] \equiv \varphi'_\varepsilon(u_\varepsilon)\partial_t u_\varepsilon \in L^2(Q_T)$ and hence it follows from (D2) that there exists a weak solution w_ε. We have that $u_\varepsilon(\cdot\,,t) = g_1(\cdot) + \int_0^t w_\varepsilon(\cdot\,,s)\,ds$ as both satisfy the same linear initial-boundary value problem. Thus $w_\varepsilon \equiv \partial_t u_\varepsilon$. Multiplying (2.16b) by $\partial_s u_\varepsilon(\cdot\,,s)$, integrating over Q_t, where s is the integration variable in time, and performing integration by parts yields that

$$\begin{aligned}
k^{-1}\int_0^t |\nabla\partial_s u_\varepsilon(\cdot\,,s)|^2_{L^2(\Omega)}\,ds &+ \int_0^t \langle[1+\varphi'_\varepsilon(u_\varepsilon(\cdot\,,s))]\partial_s u_\varepsilon(\cdot\,,s), \partial_s u_\varepsilon(\cdot\,,s)\rangle\,ds \\
&+\frac{1}{2}[k^{-1}|\partial_t u_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} + |\nabla u_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)}] \\
&= \int_0^t \langle k^{-1}\partial_s f(\cdot\,,s) + f(\cdot\,,s), \partial_s u_\varepsilon(\cdot\,,s)\rangle\,ds \\
&\quad +\frac{1}{2}[k^{-1}|\partial_t u_\varepsilon(\cdot\,,0)|^2_{L^2(\Omega)} + |\nabla u_\varepsilon(\cdot\,,0)|^2_{L^2(\Omega)}]
\end{aligned}$$

and hence

$$\begin{aligned}
k^{-1}\int_0^t |\nabla\partial_s u_\varepsilon(\cdot\,,s)|^2_{L^2(\Omega)}\,ds &+ \int_0^t \langle[1+\varphi'_\varepsilon(u_\varepsilon(\cdot\,,s))]\partial_s u_\varepsilon(\cdot\,,s), \partial_s u_\varepsilon(\cdot\,,s)\rangle\,ds \\
&+k^{-1}|\partial_t u_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} + |\nabla u_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} \\
&\le \int_0^t |k^{-1}\partial_s f(\cdot\,,s) + f(\cdot\,,s)|^2_{L^2(\Omega)}\,ds + k^{-1}|\partial_t u_\varepsilon(\cdot\,,0)|^2_{L^2(\Omega)} + |\nabla u_\varepsilon(\cdot\,,0)|^2_{L^2(\Omega)} \\
&\le \int_0^t |k^{-1}\partial_s f(\cdot\,,s) + f(\cdot\,,s)|^2_{L^2(\Omega)}\,ds + |\nabla g_1|^2_{L^2(\Omega)} \\
&\quad +2k^{-1}|\Delta g_1(\cdot) + f(\cdot\,,0)|^2_{L^2(\Omega)} + 2k|\varphi_\varepsilon(g_1) - g_2|^2_{L^2(\Omega)}. \qquad (2.17)
\end{aligned}$$

Noting (2.1c), then similar to (2.13) we have that

$$M^{-1}\varepsilon\int_0^t |\partial_s[\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))]|^2_{L^2(\Omega)}\,ds \le \int_0^t \langle\varphi'_\varepsilon(u_\varepsilon(\cdot\,,s))\partial_s u_\varepsilon(\cdot\,,s), \partial_s u_\varepsilon(\cdot\,,s)\rangle\,ds. \tag{2.18}$$

In addition we have that

$$\begin{aligned}
\frac{1}{2}k^{-1}|\partial_t v_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} &+ \int_0^t |\partial_s v_\varepsilon(\cdot\,,s)|^2_{L^2(\Omega)}\,ds \\
&= \frac{1}{2}k^{-1}|\partial_t v_\varepsilon(\cdot\,,0)|^2_{L^2(\Omega)} + \int_0^t \langle\partial_s[\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))], \partial_s v_\varepsilon(\cdot\,,s)\rangle\,ds
\end{aligned}$$

and hence that

$$\begin{aligned}
k^{-1}|\partial_t v_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)} &+ \int_0^t |\partial_s v_\varepsilon(\cdot\,,s)|^2_{L^2(\Omega)}\,ds \\
&\le k^{-1}|\partial_t v_\varepsilon(\cdot\,,0)|^2_{L^2(\Omega)} + \int_0^t |\partial_s[\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))]|^2_{L^2(\Omega)}\,ds \\
&\le k|\varphi_\varepsilon(u_\varepsilon(\cdot\,,0)) - v_\varepsilon(\cdot\,,0)|^2_{L^2(\Omega)} + \int_0^t |\partial_s[\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))]|^2_{L^2(\Omega)}\,ds. \qquad (2.19)
\end{aligned}$$

Combining (2.17)–(2.19) yields the desired result (2.15). ∎

3. A continuous in time finite element approximation. We now consider the continuous piecewise linear finite element approximation to (P_ε). Assuming (D3), we introduce

$$S^h \equiv \{\chi \in C(\bar{\Omega}^h) : \chi|_\kappa \text{ is linear for all } \kappa \in T^h\}$$
$$\text{and} \qquad S_0^h \equiv \{\chi \in S^h : \chi = 0 \text{ on } \partial\Omega^h\}.$$

For the purposes of the analysis we extend $\chi \in S^h$ from $\bar{\Omega}^h$ to $\bar{\Omega} \setminus \bar{\Omega}^h$ by zero. Let $\pi_h : C^0(\bar{\Omega}) \to S^h$ denote the interpolation operator such that for any $w \in C^0(\bar{\Omega})$, $\pi_h w \in S^h$ satisfies

$$(\pi_h w)(x_i) = w(x_i) \quad \text{for all nodes } x_i \text{ of the partition } T^h.$$

Let $P_h^0 : L^2(\Omega) \to S^h$ denote the L^2 projection such that for any $w \in L^2(\Omega)$, $P_h^0 w \in S^h$ satisfies

$$\langle w - P_h^0 w, \chi\rangle = 0 \quad \forall \chi \in S^h.$$

Let $P_h^1 : H_0^1 \to S_0^h$ denote the H^1-seminorm projection such that for any $w \in H_0^1(\Omega)$, $P_h^1 w \in S_0^h$ satisfies

$$\langle \nabla(w - P_h^1 w), \nabla\chi\rangle = 0 \quad \forall \chi \in S_0^h.$$

We recall the standard approximation results for all $\kappa \in T^h$,

$$|w - \pi_h w|_{W^{m,q}(\kappa)} \le C h_\kappa^{2-m} |w|_{W^{2,q}(\kappa)} \quad \text{for } m = 0 \text{ and } 1 \text{ and}$$
$$\forall q \in [1,\infty] \text{ if } d \le 2 \text{ and } \forall q \in \left(\tfrac{3}{2},\infty\right] \text{ if } d = 3, \tag{3.1a}$$
$$|w - P_h^0 w|_{L^2(\Omega)} \le C h^m |w|_{H^m(\Omega)} \quad \text{for } m = 0,\ 1, \text{ and } 2 \tag{3.1b}$$

and

$$|w - P_h^1 w|_{L^2(\Omega)} + h|w - P_h^1 w|_{H^1(\Omega)} \le C h^m |w|_{H^m(\Omega)} \quad \text{for } m = 1 \text{ and } 2, \tag{3.1c}$$

where in (3.1a) we note the imbedding $W^{2,1}(\kappa) \subset C^0(\bar{\kappa})$ in the case $d = 2$; see, e.g., [12, p. 300] and for the "skin" effects in (3.1b, 3.1c), see, e.g., [1, Lem. 3.2]. Another result that will be useful later is that

$$|(I - \pi_h)\varphi_\varepsilon(\chi)|_{L^2(\Omega)} \le C h |\nabla \pi_h[\varphi_\varepsilon(\chi)]|_{L^2(\Omega)} \quad \forall \chi \in S_0^h; \tag{3.2}$$

see [8, p. 68].

The standard Galerkin approximation to (P_ε) is then as follows.

$(\mathbf{P}_\varepsilon^h)$ Find $u_\varepsilon^h \in H^1(0,T;S_0^h)$ and $v_\varepsilon^h \in H^1(0,T;S^h)$ such that

$$\langle \partial_t u_\varepsilon^h + \partial_t v_\varepsilon^h, \chi\rangle + \langle \nabla u_\varepsilon^h, \nabla\chi\rangle = \langle f, \chi\rangle \quad \forall \chi \in S_0^h,$$
$$\langle \partial_t v_\varepsilon^h, \chi\rangle = k\langle \varphi_\varepsilon(u_\varepsilon^h) - v_\varepsilon^h, \chi\rangle \quad \forall \chi \in S^h,$$
$$u_\varepsilon^h(\cdot\,,0) = P_h^1 g_1(\cdot) \qquad v_\varepsilon^h(\cdot\,,0) = P_h^0 g_2(\cdot).$$

The above approximation is not practical as it requires the term $\langle \varphi_\varepsilon(u_\varepsilon^h), \chi\rangle$ to be integrated exactly. This is obviously difficult in practice and it is computationally more convenient to consider a scheme where numerical integration is applied to all the terms and the initial data are interpolated as opposed to being projected. Below we introduce and analyze such a scheme.

For all $w_1,\ w_2 \in C^0(\bar{\Omega}^h)$ we set

$$\langle w_1, w_2 \rangle^h \equiv \int_{\Omega^h} \pi_h(w_1, w_2)$$

as an approximation to $\langle w_1, w_2 \rangle$. On setting

$$|w|_h \equiv [\langle w, w \rangle^h]^{\frac{1}{2}} \quad \text{for } w \in C^0(\bar{\Omega}^h),$$

we recall the well-known results

$$|\chi|_{L^2(\Omega^h)} \le |\chi|_h \le C_1 |\chi|_{L^2(\Omega^h)} \quad \forall \chi \in S^h \tag{3.3a}$$

and for $m = 0$ or 1,

$$\left| \int_{\Omega^h} \chi_1 \chi_2 - \langle \chi_1, \chi_2 \rangle^h \right| \le C_2 h^{1+m} \|\chi_1\|_{H^1(\Omega^h)} \|\chi_2\|_{H^m(\Omega^h)} \quad \forall \chi_1, \chi_2 \in S^h. \tag{3.3b}$$

Assuming (D4), a more practical approximation to (P_ε) than $(\mathrm{P}_\varepsilon^h)$ is then as follows:

$(\hat{\mathbf{P}}_\varepsilon^h)$ Find $\hat{u}_\varepsilon^h \in H^1(0,T;S_0^h)$ and $\hat{v}_\varepsilon^h \in H^1(0,T;S^h)$ such that

$$\begin{aligned}
\langle \partial_t \hat{u}_\varepsilon^h + \partial_t \hat{v}_\varepsilon^h, \chi \rangle^h + \langle \nabla \hat{u}_\varepsilon^h, \nabla \chi \rangle &= \langle f, \chi \rangle^h \quad \forall \chi \in S_0^h, \\
\langle \partial_t \hat{v}_\varepsilon^h, \chi \rangle^h &= k \langle \varphi_\varepsilon(\hat{u}_\varepsilon^h) - \hat{v}_\varepsilon^h, \chi \rangle^h \quad \forall \chi \in S^h, \\
\hat{u}_\varepsilon^h(\cdot\,,0) = g_1^h(\cdot) \qquad & \hat{v}_\varepsilon^h(\cdot\,,0) = g_2^h(\cdot),
\end{aligned}$$

where $g_1^h \in S_0^h$ and $g_2^h \in S^h$ are suitable approximations to g_1 and g_2 satisfying

$$|\langle \nabla g_1^h, \nabla \chi \rangle| \le C |\chi|_{L^2(\Omega)} \quad \forall \chi \in S_0^h \qquad \text{and} \qquad |g_2^h|_{L^2(\Omega)} \le C. \tag{3.4a}$$

The first inequality in (3.4a) yields that

$$|g_1^h|_{H^1(\Omega)} \le C \qquad \text{and hence that} \qquad |\varphi_\varepsilon(g_1^h)|_h \le C. \tag{3.4b}$$

For the main error bounds in this paper, we will take $g_i^h \equiv \pi_h g_i$, i.e., the most practical choice. However, for the purposes of part 2 we will develop many of the preliminary results for general initial data satisfying (3.4a). We note that (3.1a) and the bound $|\nabla \chi|_{L^2(\kappa)} \le C h_\kappa^{-1} |\chi|_{L^2(\kappa)}$ yield that

$$|\langle \nabla \pi_h g_1, \nabla \chi \rangle| = |\langle \nabla (I - \pi_h) g_1, \nabla \chi \rangle + \langle \Delta g_1, \chi \rangle| \le C |g_1|_{H^2(\Omega)} |\chi|_{L^2(\Omega)} \tag{3.4c}$$

and hence the bounds (3.4a) hold for $g_i^h \equiv \pi_h g_i$, $i = 1, 2$.

THEOREM 3.1. *Let the assumptions* (D4) *hold. Then for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $h > 0$ *there exists a unique solution* $\{\hat{u}_\varepsilon^h, \hat{v}_\varepsilon^h\}$ *to* $(\hat{\mathrm{P}}_\varepsilon^h)$ *and* $|\hat{u}_\varepsilon^h|_{L^\infty(Q_T)}, |\hat{v}_\varepsilon^h|_{L^\infty(Q_T)} \le C(k,h)$.

Proof. Existence and uniqueness of a solution follow from standard ordinary differential equation theory using bounds similar to those established in Theorem 2.1 and exploiting the equivalence of norms on S^h. ∎

LEMMA 3.1. *Under assumptions* (D4) *we have for all* $\varepsilon \in (0, \varepsilon_0]$, $h > 0$ *and* $t \in (0, T]$ *that*

$$\begin{aligned}
&|\nabla \hat{u}_\varepsilon^h(\cdot\,,t)|^2_{L^2(\Omega)} + |\partial_t \hat{u}_\varepsilon^h|^2_{L^2(Q_T)} + \varepsilon |\partial_t \hat{v}_\varepsilon^h|^2_{L^2(Q_T)} + \varepsilon |\partial_t \pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_T)} \\
&\quad + k^{-1}[|\partial_t \hat{u}_\varepsilon^h(\cdot\,,t)|^2_{L^2(\Omega)} + \varepsilon |\partial_t \hat{v}_\varepsilon^h(\cdot\,,t)|^2_{L^2(\Omega)} + |\nabla(\partial_t \hat{u}_\varepsilon^h)|^2_{L^2(Q_T)}] \\
&\le C \left[1 + k |\varphi_\varepsilon(g_1^h) - g_2^h|_h^2 \right] \le Ck. \qquad (3.5)
\end{aligned}$$

Proof. This is a discrete analogue of Lemma 2.2, after noting (3.4); see [2]. □

In order to analyze the approximation $(\hat{\mathrm{P}}^h_\varepsilon)$ it is convenient to introduce an associated linear problem of $(\mathrm{P}^h_\varepsilon)$; see Remark 3.2.

$(\mathbf{P}^{h,*}_\varepsilon)$ Find $u^{h,*}_\varepsilon \in H^1(0,T;S^h_0)$ and $v^{h,*}_\varepsilon \in H^1(0,T;S^h)$ such that

$$\langle \partial_t u^{h,*}_\varepsilon + \partial_t v^{h,*}_\varepsilon, \chi\rangle + \langle \nabla u^{h,*}_\varepsilon, \nabla\chi\rangle = \langle f,\chi\rangle \quad \forall \chi \in S^h_0$$
$$\langle \partial_t v^{h,*}_\varepsilon, \chi\rangle = k\langle \varphi_\varepsilon(u_\varepsilon) - v^{h,*}_\varepsilon, \chi\rangle \quad \forall\chi\in S^h$$
$$u^{h,*}_\varepsilon(\cdot\,,0) = P^1_h g_1(\cdot) \qquad v^{h,*}_\varepsilon(\cdot\,,0) = P^0_h g_2(\cdot).$$

The existence and uniqueness of $\{u^{h,*}_\varepsilon, v^{h,*}_\varepsilon\}$ solving $(\mathrm{P}^{h,*}_\varepsilon)$ for all $\varepsilon\in(0,\varepsilon_0]$ and $h>0$ are easily established under assumptions (D3), and we have the following result.

LEMMA 3.2. *Assuming* (D3), *we have for all* $\varepsilon\in(0,\varepsilon_0]$, $h>0$ *and* $t\in(0,T]$ *that*

$$|u_\varepsilon - u^{h,*}_\varepsilon|^2_{L^2(Q_t)} + h^2\left|\nabla\int_0^t (u_\varepsilon - u^{h,*}_\varepsilon)(\cdot\,,s)\,ds\right|^2_{L^2(\Omega)}$$
$$\le Ch^4\left[|u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))} + |g_1|^2_{H^2(\Omega)}\right] \le Ckh^4, \tag{3.6a}$$
$$|(u_\varepsilon - u^{h,*}_\varepsilon)(\cdot\,,t)|^2_{L^2(\Omega)}$$
$$\le Ch^2\left[|u_\varepsilon|^2_{H^1(0,t;H^1(\Omega))} + |\nabla u_\varepsilon(\cdot\,,t)|^2_{L^2(\Omega)}\right] \le Ck^2h^2, \tag{3.6b}$$
$$|\nabla(u_\varepsilon - u^{h,*}_\varepsilon)|^2_{L^2(Q_t)}$$
$$\le Ch^2\left[|u_\varepsilon|^2_{H^1(0,t;H^1(\Omega))} + |u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))}\right] \le Ck^2h^2, \tag{3.6c}$$

and if $g_2\in H^1(\Omega)$

$$\|v_\varepsilon - v^{h,*}_\varepsilon\|^2_{E_1(k,t)} \le Ch^2\left[|\nabla\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} + k^{-1}|g_2|^2_{H^1(\Omega)}\right] \le C\varepsilon^{-1}h^2. \tag{3.6d}$$

Proof. The problem (P_ε) can be restated as find $u_\varepsilon(x,t)$ such that

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = f + F_\varepsilon(t,u_\varepsilon) \quad \text{in } Q_T, \tag{3.7a}$$
$$u_\varepsilon = 0 \quad \text{on } \partial\Omega\times(0,T] \qquad u_\varepsilon(\cdot\,,0) = g_1(\cdot) \quad \text{in } \Omega, \tag{3.7b}$$

where

$$F_\varepsilon(t,w(\cdot\,,t)) \equiv k\left[e^{-kt}g_2 - \varphi_\varepsilon(w(\cdot\,,t)) + k\int_0^t e^{-k(t-s)}\varphi_\varepsilon(w(\cdot\,,s))\,ds\right]. \tag{3.7c}$$

Similarly, $(\mathrm{P}^{h,*}_\varepsilon)$ can be restated as find $u^{h,*}_\varepsilon\in H^1(0,T;S^h_0)$ such that

$$\langle\partial_t u^{h,*}_\varepsilon,\chi\rangle + \langle\nabla u^{h,*}_\varepsilon,\nabla\chi\rangle = \langle f + F_\varepsilon(t,u_\varepsilon),\chi\rangle \quad \forall\chi\in S^h_0,$$
$$u^{h,*}_\varepsilon(\cdot\,,0) = P^1_h g_1(\cdot).$$

Let $e^{h,*}_{u,\varepsilon} \equiv u_\varepsilon - u^{h,*}_\varepsilon$ and so we have that

$$\langle\partial_t e^{h,*}_{u,\varepsilon},\chi\rangle + \langle\nabla e^{h,*}_{u,\varepsilon},\nabla\chi\rangle = 0 \quad \forall\chi\in S^h_0, \tag{3.8a}$$
$$e^{h,*}_{u,\varepsilon}(\cdot\,,0) = g_1(\cdot) - P^1_h g_1(\cdot). \tag{3.8b}$$

With $e^{h,*}_{u,\varepsilon} \equiv (u_\varepsilon - P^1_h u_\varepsilon) + (P^1_h u_\varepsilon - u^{h,*}_\varepsilon) \equiv \rho + \vartheta$, it follows by choosing $\chi \equiv \int_s^t \vartheta(\cdot\,,\sigma)\,d\sigma$, integrating over $(0,t)$ in time, where s is the integration variable in time, and performing integration by parts yields that

$$\int_0^t |\vartheta(\cdot\,,s)|^2_{L^2(\Omega)}\,ds + \frac{1}{2}\left|\nabla\int_0^t\vartheta(\cdot\,,s)\,ds\right|^2_{L^2(\Omega)}$$
$$= -\int_0^t\langle\rho(\cdot\,,s),\vartheta(\cdot\,,s)\rangle\,ds + \left\langle g_1(\cdot) - P^1_h g_1(\cdot), \int_0^t\vartheta(\cdot\,,s)\,ds\right\rangle. \tag{3.9}$$

Under the stated assumptions on Ω we have from (P_ε) that $u_\varepsilon \in L^2(0,T;H^2(\Omega))$ and

$$\|u_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq C\left[|\partial_t u_\varepsilon|_{L^2(Q_T)} + |\partial_t v_\varepsilon|_{L^2(Q_T)} + |f|_{L^2(Q_T)}\right] \leq Ck^{\frac{1}{2}}, \tag{3.10}$$

where we have noted (2.11) and (2.13). From (3.9), (3.1c), and (3.10) we have that

$$\begin{aligned}|e^{h,*}_{u,\varepsilon}|^2_{L^2(Q_t)} &\leq C\left[|\vartheta|^2_{L^2(Q_t)} + |\rho|^2_{L^2(Q_t)}\right] \leq C|\rho|^2_{L^2(Q_t)} + Ch^4|g_1|^2_{H^2(\Omega)} \\ &\leq Ch^4\left[|u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))} + |g_1|^2_{H^2(\Omega)}\right] \leq Ckh^4\end{aligned}$$

and

$$\begin{aligned}\left|\nabla\int_0^t e^{h,*}_{u,\varepsilon}(\cdot\,,s)\,ds\right|^2_{L^2(\Omega)} &\leq C\left[\left|\nabla\int_0^t \rho(\cdot\,,s)\,ds\right|^2_{L^2(\Omega)} + |\rho|^2_{L^2(Q_t)}\right] + Ch^2|g_1|^2_{H^1(\Omega)} \\ &\leq Ch^2\left[|u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))} + |g_1|^2_{H^1(\Omega)}\right] \leq Ckh^2.\end{aligned}$$

Hence we obtain (3.6a).

In addition, choosing $\chi \equiv \vartheta$ in (3.8a) yields that

$$|\vartheta(\cdot\,,t)|^2_{L^2(\Omega)} + |\nabla\vartheta|^2_{L^2(Q_t)} \leq C\int_0^t |\partial_s\rho(\cdot\,,s)|^2_{L^2(\Omega)}\,ds.$$

Hence from (3.1c), (2.15), and (3.9) we obtain the results (3.6b, 3.6c). Finally, if $g_2 \in H^1(\Omega)$, setting $e^{h,*}_{v,\varepsilon} \equiv v_\varepsilon - v^{h,*}_\varepsilon$ we have from (3.1b) and (2.11) that

$$\begin{aligned}\|e^{h,*}_{v,\varepsilon}\|^2_{E_1(k,t)} &= \int_0^t \left\langle (I-P^0_h)\varphi_\varepsilon(u_\varepsilon(\cdot\,,s)), e^{h,*}_{v,\varepsilon}(\cdot\,,s)\right\rangle ds + \frac{1}{2}k^{-1}|(I-P^0_h)g_2|^2_{L^2(\Omega)} \\ &\leq Ch^2\left[|\nabla\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} + k^{-1}|g_2|^2_{H^1(\Omega)}\right] \leq C\varepsilon^{-1}h^2.\end{aligned}$$

Hence the desired result (3.6d). □

Let $B \equiv \{b_{ij}\}^I_{i,j=1} \equiv \{\langle\nabla\chi_i, \nabla\chi_j\rangle\}^I_{i,j=1}$ and $A \equiv \{\langle\chi_i,\chi_j\rangle^h\}^I_{i,j=1}$, where $\{x_j\}^I_{j=1}$ are the internal nodes, $\{x_j\}^J_{j=I+1}$ are the boundary nodes of the partitioning T^h, and $\chi_j \in S^h$ is such that $\chi_j(x_i) = \delta_{ij}$, $i,j = 1 \to J$. It follows that A is a diagonal matrix with positive entries and that B and $\tilde{B} \equiv A^{-1}B$ are positive definite. Under assumption (D5) it follows that $b_{ij} \leq 0$ for $i \neq j$ and hence B and $\tilde{B}$ are M-matrices. From this property one can deduce the discrete analogue of (2.13)

$$M^{-1}\varepsilon|\nabla\pi_h[\varphi_\varepsilon(\chi)]|^2_{L^2(\Omega)} \leq \langle\nabla\chi, \nabla\pi_h[\varphi_\varepsilon(\chi)]\rangle \quad \forall\chi \in S^h_0; \tag{3.11}$$

see, e.g., [13, section 2.4.2]. Furthermore, it follows from (3.11), (3.1a), (2.3a), and (3.2) that for all $w \in H^1_0(\Omega)\cap H^2(\Omega)$,

$$\begin{aligned}M^{-1}\varepsilon|\nabla\pi_h[\varphi_\varepsilon(w)]|^2_{L^2(\Omega)} &\leq \langle\nabla\pi_h w, \nabla\pi_h[\varphi_\varepsilon(w)]\rangle \\ &= \langle\nabla w, \nabla\varphi_\varepsilon(w)\rangle + \langle\Delta w, (I-\pi_h)\varphi_\varepsilon(w)\rangle - \langle\nabla(I-\pi_h)w, \nabla\pi_h[\varphi_\varepsilon(w)]\rangle \\ &\leq \langle\nabla w, \nabla\varphi_\varepsilon(w)\rangle + \langle\Delta w, [\varphi_\varepsilon(w) - \varphi_\varepsilon(\pi_h w)] + (I-\pi_h)\varphi_\varepsilon(\pi_h w)\rangle \\ &\qquad + C\varepsilon^{-1}h^2|w|^2_{H^2(\Omega)} \\ &\leq \langle\nabla w, \nabla\varphi_\varepsilon(w)\rangle + C\varepsilon^{-1}h^2|w|^2_{H^2(\Omega)}. \qquad (3.12)\end{aligned}$$

We have the following corollary to Theorem 3.1.

COROLLARY 3.1. *Let the assumptions* (D5) *hold. Then the unique solution* $\{\hat{u}^h_\varepsilon, \hat{v}^h_\varepsilon\}$ *to* $(\hat{\mathrm{P}}^h_\varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$ *is such that*

$$\underline{u} \le \hat{u}^h_\varepsilon \le \overline{u} \quad \textit{and} \quad \underline{v} \le \hat{v}^h_\varepsilon \le \overline{v} \quad \textit{in } Q_T, \tag{3.13}$$

where $\underline{u}$, $\overline{u}$, $\underline{v}$, *and* $\overline{v} \in C^0[0,T]$ *are the bounds from Theorem* 2.1 *with* g_i *replaced by* g^h_i, $i = 1, 2$. *In particular if* g^h_1, g^h_2, *and* $f \ge 0$ *then one can take* $\underline{u} = \underline{v} = 0$.

Proof. This is a discrete analogue of Theorem 2.1; see [2]. □

LEMMA 3.3. *Under assumptions* (D5) *we have for all* $\varepsilon \in (0, \varepsilon_0]$ *and* $h > 0$, *provided* $\varepsilon^{-1} k h^2 \le C$, *and for all* $t \in (0, T]$ *that*

$$\begin{aligned} &\|u^{h,*}_\varepsilon - \hat{u}^h_\varepsilon\|^2_{E_2(k,t)} + \varepsilon \left|\pi_h \left[\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(\hat{u}^h_\varepsilon)\right]\right|^2_{L^2(Q_t)} \\ &\quad \le C \left[\varepsilon^{-1} + k|\varphi_\varepsilon(g^h_1) - g^h_2|^2_h + \|g^h_1\|^2_{H^1(\Omega)}\right] h^2 \\ &\quad\quad + C \left[|P^1_h g_1 - g^h_1|^2_{L^2(\Omega)} + \sup_{\chi \in S^h_0} \left\{ \frac{|\langle g_2, \chi\rangle - \langle g^h_2, \chi\rangle^h|}{\|\chi\|_{H^1(\Omega)}} \right\}^2 \right] \end{aligned} \tag{3.14a}$$

and

$$\varepsilon \|v^{h,*}_\varepsilon - \hat{v}^h_\varepsilon\|^2_{E_1(k,t)} \le C \left[h^2 + \varepsilon \left|\pi_h[\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(\hat{u}^h_\varepsilon)]\right|^2_{L^2(Q_t)} + \varepsilon k^{-1} |g_2 - g^h_2|^2_{L^2(\Omega)}\right]. \tag{3.14b}$$

Proof. Let $\hat{e}^{u,h}_\varepsilon \equiv u^{h,*}_\varepsilon - \hat{u}^h_\varepsilon$ and $\hat{e}^{v,h}_\varepsilon \equiv v^{h,*}_\varepsilon - \hat{v}^h_\varepsilon$. Subtracting the first equation in $(\hat{\mathrm{P}}^h_\varepsilon)$ from that in $(\mathrm{P}^{h,*}_\varepsilon)$, choosing $\chi \equiv \int_s^t \hat{e}^{u,h}_\varepsilon(\cdot, \sigma)\, d\sigma$, integrating over $(0, t)$ in time, where s is the integration variable in time, and performing integration by parts yields that

$$\begin{aligned} \int_0^t |\hat{e}^{u,h}_\varepsilon(\cdot, s)|^2_{L^2(\Omega)}\, ds + \frac{1}{2} \left|\nabla \int_0^t \hat{e}^{u,h}_\varepsilon(\cdot, s)\, ds\right|^2_{L^2(\Omega)} &= -\int_0^t \langle \hat{e}^{v,h}_\varepsilon(\cdot, s), \hat{e}^{u,h}_\varepsilon(\cdot, s)\rangle\, ds \\ + \left\langle \hat{e}^{u,h}_\varepsilon(\cdot, 0) + g_2(\cdot), \int_0^t \hat{e}^{u,h}_\varepsilon(\cdot, s)\, ds \right\rangle &- \left\langle g^h_2(\cdot), \int_0^t \hat{e}^{u,h}_\varepsilon(\cdot, s)\, ds \right\rangle^h \\ + \int_0^t \left[\langle \xi(\cdot, s), \hat{e}^{u,h}_\varepsilon(\cdot, s)\rangle - \langle \xi(\cdot, s), \hat{e}^{u,h}_\varepsilon(\cdot, s)\rangle^h\right] ds&, \end{aligned} \tag{3.15a}$$

where

$$\begin{aligned} \xi(\cdot, t) &\equiv \int_0^t (f - \partial_s \hat{u}^h_\varepsilon - \partial_s \hat{v}^h_\varepsilon)(\cdot, s)\, ds - \hat{v}^h_\varepsilon(\cdot, 0) \\ &= g^h_1(\cdot) - (\hat{u}^h_\varepsilon + \hat{v}^h_\varepsilon)(\cdot, t) + \int_0^t f(\cdot, s)\, ds. \end{aligned} \tag{3.15b}$$

Similarly, choosing $\chi \equiv \hat{e}^{u,h}_\varepsilon$ yields that

$$\begin{aligned} &\frac{1}{2} |\hat{e}^{u,h}_\varepsilon(\cdot, t)|^2_{L^2(\Omega)} + \int_0^t |\nabla \hat{e}^{u,h}_\varepsilon(\cdot, s)|^2_{L^2(\Omega)}\, ds \\ &\quad = -\int_0^t \left\langle \partial_s \hat{e}^{v,h}_\varepsilon(\cdot, s), \hat{e}^{u,h}_\varepsilon(\cdot, s)\right\rangle\, ds + \frac{1}{2} |\hat{e}^{u,h}_\varepsilon(\cdot, 0)|^2_{L^2(\Omega)} \\ &\quad\quad + \int_0^t \left[\langle \partial_s \xi(\cdot, s), \hat{e}^{u,h}_\varepsilon(\cdot, s)\rangle - \langle \partial_s \xi(\cdot, s), \hat{e}^{u,h}_\varepsilon(\cdot, s)\rangle^h\right] ds. \end{aligned} \tag{3.16}$$

Therefore from (3.15), (3.16), and the second equations in ($\mathrm{P}_\varepsilon^{h,*}$) and ($\hat{\mathrm{P}}_\varepsilon^h$), it follows that

$$
\begin{aligned}
&\|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)} + \int_0^t \left\langle \varphi_\varepsilon(u_\varepsilon(\cdot\,,s)) - \varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot\,,s)), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right\rangle^h ds \\
&\quad = \int_0^t \left[\left\langle(1+k^{-1}\partial_s)\xi(\cdot\,,s), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right\rangle - \left\langle(1+k^{-1}\partial_s)\xi(\cdot\,,s), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right\rangle^h\right] ds \\
&\qquad - \int_0^t \left\langle(1+k^{-1}\partial_s)\hat{e}_\varepsilon^{v,h}(\cdot\,,s), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right\rangle ds \\
&\qquad + \int_0^t \left\langle \varphi_\varepsilon(u_\varepsilon(\cdot\,,s)) - \varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot\,,s)), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right\rangle^h ds + \frac{1}{2}k^{-1}|\hat{e}_\varepsilon^{u,h}(\cdot\,,0)|^2_{L^2(\Omega)} \\
&\qquad + \left\langle \hat{e}_\varepsilon^{u,h}(\cdot\,,0) + g_2(\cdot), \int_0^t \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\,ds\right\rangle - \left\langle g_2^h(\cdot), \int_0^t \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\,ds\right\rangle^h \\
&\quad = \Bigg[\left\langle \hat{e}_\varepsilon^{u,h}(\cdot\,,0) + g_2(\cdot), \int_0^t \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\,ds\right\rangle - \left\langle g_2^h(\cdot), \int_0^t \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\,ds\right\rangle^h \\
&\qquad + \frac{1}{2}k^{-1}|\hat{e}_\varepsilon^{u,h}(\cdot\,,0)|^2_{L^2(\Omega)}\Bigg] + \int_0^t \left[\langle \eta(\cdot\,,s), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\rangle - \langle \eta(\cdot\,,s), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\rangle^h\right] ds \\
&\quad \equiv T_1 + T_2,
\end{aligned} \tag{3.17a}
$$

where

$$
\begin{aligned}
\eta(\cdot\,,t) &\equiv (1+k^{-1}\partial_t)(\xi + \hat{v}_\varepsilon^h)(\cdot\,,t) - \varphi_\varepsilon(u_\varepsilon(\cdot\,,t)) \\
&\equiv \eta_1(\cdot\,,t) + \eta_2(\cdot\,,t) + \eta_3(\cdot\,,t),
\end{aligned} \tag{3.17b}
$$

$$
\eta_1(\cdot\,,t) \equiv g_1^h(\cdot) - (1+k^{-1}\partial_t)\hat{u}_\varepsilon^h(\cdot\,,t) \in S^h, \tag{3.17c}
$$

$$
\eta_2(\cdot\,,t) \equiv (1+k^{-1}\partial_t)\int_0^t f(\cdot\,,s)\,ds, \tag{3.17d}
$$

and

$$
\eta_3(\cdot\,,t) \equiv -\varphi_\varepsilon(u_\varepsilon(\cdot\,,t)). \tag{3.17e}
$$

It follows that

$$
\begin{aligned}
T_1 &\le \left\langle (P_h^1 g_1 - g_1^h) + g_2, \int_0^t \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\,ds\right\rangle - \left\langle g_2^h(\cdot), \int_0^t \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\,ds\right\rangle^h \\
&\quad + C|P_h^1 g_1 - g_1^h|^2_{L^2(\Omega)} \\
&\le C\left[|P_h^1 g_1 - g_1^h|^2_{L^2(\Omega)} + \sup_{\chi\in S_0^h}\left\{\frac{|\langle g_2,\chi\rangle - \langle g_2^h,\chi\rangle^h|}{\|\chi\|_{H^1(\Omega)}}\right\}^2\right] + \frac{1}{2}\|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)}.
\end{aligned} \tag{3.18}
$$

Next we note that

$$
\begin{aligned}
T_2 &\equiv \int_0^t \left[\langle \pi_h\eta(\cdot\,,s), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\rangle - \langle \pi_h\eta(\cdot\,,s), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\rangle^h\right] ds \\
&\quad + \int_0^t \left\langle [(I-\pi_h)(\eta_2+\eta_3)](\cdot\,,s), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right\rangle ds \equiv T_{2,1} + T_{2,2}.
\end{aligned} \tag{3.19}
$$

From (3.12), (2.11), (3.10), and the assumption $\varepsilon^{-1}kh^2 \le C$ we have that

$$
\varepsilon|\pi_h[\varphi_\varepsilon(u_\varepsilon)]|^2_{L^2(0,T;H^1(\Omega))} \le C \tag{3.20a}
$$

and, in addition, noting (2.3a) and (3.2),

$$|(I-\pi_h)\varphi_\varepsilon(u_\varepsilon)|_{L^2(Q_T)} \leq |\varphi_\varepsilon(u_\varepsilon)-\varphi_\varepsilon(\pi_h u_\varepsilon)|_{L^2(Q_T)} + |(I-\pi_h)\varphi_\varepsilon(\pi_h u_\varepsilon)|_{L^2(Q_T)}$$
$$\leq C\varepsilon^{-\frac{1}{2}}h. \tag{3.20b}$$

Therefore we have from (3.3b), (3.1a), (3.5), (3.20), and (3.2) that

$$\begin{aligned} T_{2,1} &\leq Ch\int_0^t \|\pi_h\eta(\cdot\,,s)\|_{H^1(\Omega)}|\hat{e}_\varepsilon^{u,h}(\cdot\,,s)|_{L^2(\Omega)}\,ds \\ &\leq Ch\left[\int_0^t \|\pi_h\eta(\cdot\,,s)\|^2_{H^1(\Omega)}\,ds\right]^{\frac{1}{2}}\|\hat{e}_\varepsilon^{u,h}\|_{E_2(k,t)} \\ &\leq C\left[\varepsilon^{-1}+k|\varphi_\varepsilon(g_1^h)-g_2^h|_h^2+\|g_1^h\|^2_{H^1(\Omega)}\right]h^2 \end{aligned} \tag{3.21a}$$

and

$$\begin{aligned} T_{2,2} &\leq \int_0^t |(I-\pi_h)(\eta_2+\eta_3)(\cdot\,,s)|_{L^2(\Omega)}\left|\hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right|_{L^2(\Omega)}\,ds \\ &\leq C\varepsilon^{-\frac{1}{2}}h\|\hat{e}_\varepsilon^{u,h}\|_{E_2(k,t)}. \end{aligned} \tag{3.21b}$$

It follows from (3.3a), (2.3a), (3.1a), and (3.6a) that

$$\begin{aligned} &\|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)} + M^{-1}\varepsilon\left|\pi_h[\varphi_\varepsilon(u_\varepsilon)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|^2_{L^2(Q_t)} \\ &\quad\leq \|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)} + \int_0^t\left\langle\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))-\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot\,,s)),u_\varepsilon-\hat{u}_\varepsilon^h\right\rangle^h ds \\ &\quad\leq \|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)} + \int_0^t\left\langle\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))-\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot\,,s)),\hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right\rangle^h ds \\ &\qquad+C\varepsilon^{-1}|\pi_h u_\varepsilon-u_\varepsilon^{h,*}|^2_{L^2(Q_t)} \\ &\quad\leq \|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)} + \int_0^t\left\langle\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))-\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot\,,s)),\hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right\rangle^h ds + C\varepsilon^{-1}kh^4. \end{aligned} \tag{3.22}$$

Combining (3.17a), (3.18), (3.19), (3.21), and (3.22) yields the desired result (3.14a).

Finally, we have from (3.20b) that

$$\begin{aligned} &\|\hat{e}_\varepsilon^{v,h}\|^2_{E_1(k,t)} \\ &\quad= \int_0^t\left\langle\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))-\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot\,,s))],\hat{e}_\varepsilon^{v,h}(\cdot\,,s)\right\rangle ds + \frac{1}{2}k^{-1}|P_h^0 g_2-g_2^h|^2_{L^2(\Omega)} \\ &\quad\leq C\left[\left|\pi_h[\varphi_\varepsilon(u_\varepsilon)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|^2_{L^2(Q_t)} + |(I-\pi_h)[\varphi_\varepsilon(u_\varepsilon)]|^2_{L^2(Q_t)} + k^{-1}|g_2-g_2^h|^2_{L^2(\Omega)}\right] \\ &\quad\leq C\left[\left|\pi_h[\varphi_\varepsilon(u_\varepsilon)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|^2_{L^2(Q_t)} + \varepsilon^{-1}h^2 + k^{-1}|g_2-g_2^h|^2_{L^2(\Omega)}\right] \end{aligned} \tag{3.23}$$

and hence the desired result (3.14b). □

We now improve on the bounds (3.14a, 3.14b) in the physically interesting case of given data g_1, g_2, and $f \geq 0$ yielding u, $u_\varepsilon \geq 0$ in Q_T (and $\hat{u}_\varepsilon^h \geq 0$ in Q_T if $g_i^h \equiv \pi_h g_i$, $i=1,2$). Assuming (D6), we set φ_ε to be the following quadratic regularization of φ

$$\varphi_\varepsilon(s) \equiv \begin{cases} as^2+bs & \text{for } s\in[0,\delta], \\ \varphi(s) & \text{otherwise,} \end{cases} \tag{3.24}$$

where $a \equiv \delta^{-1}\varphi'(\delta) - \delta^{-2}\varphi(\delta)$, $b \equiv -\varphi'(\delta) + 2\delta^{-1}\varphi(\delta)$, and $\delta \equiv \varepsilon^{1/(1-p)}$ so that $\varphi_\varepsilon \in C^1[0,\infty)$. From (D6)(iii) it follows that $\varphi(\delta) \geq \delta\varphi'(\delta)$, which in turn yields that $0 < b \leq C_1\varepsilon^{-1}$ and $-C_2\varepsilon^{(p-2)/(1-p)} \leq a \leq 0$, see (2.3b), and hence φ_ε satisfies the conditions (2.1). Therefore all the results proven so far in this paper hold under the assumptions (D6). We note for example that $\varphi(s) \equiv \alpha s^p$ for $s \geq 0$ with $p \in (0,1)$ and $\alpha \in \boldsymbol{R}^+$ satisfies (1.9) and (D6)(iii).

Assuming (D6), it follows for all $w \in H_0^1(\Omega) \cap W^{2,1}(\Omega)$ with $w(x) \in [0,m]$ for $x \in \Omega$ that

$$\begin{aligned} |\varphi_\varepsilon(w)|_{W^{2,1}(\Omega)} &\leq C\left[\left|\varphi_\varepsilon''(w)|\nabla w|^2\right|_{L^1(\Omega)} + \varepsilon^{-1}|w|_{W^{2,1}(\Omega)}\right] \\ &\leq C\varepsilon^{-1}\|w\|_{W^{2,1}(\Omega)}. \end{aligned} \tag{3.25}$$

Since $\varphi_\varepsilon''(s) \leq 0$ for almost all $s \in [0,m]$, Theorem 7.8 and section 7.5 in [9] and 5.12.5 in [12] yield that

$$\begin{aligned} \left|\varphi_\varepsilon''(w)|\nabla w|^2\right|_{L^1(\Omega)} &\equiv |\langle\varphi_\varepsilon''(w)\nabla w, \nabla w\rangle| = |\langle\nabla[\varphi_\varepsilon'(w)], \nabla w\rangle| \\ &\leq |\langle\varphi_\varepsilon'(w), \Delta w\rangle| + \left|\int_{\partial\Omega}\varphi_\varepsilon'(w)\nabla w.\underline{n}\right| \leq C\varepsilon^{-1}\|w\|_{W^{2,1}(\Omega)}. \end{aligned}$$

From (D6)(i) we have the discrete Sobolev imbedding

$$|\chi|_{L^\infty(\Omega)} \leq C[\ln(1/h)]^r\|\chi\|_{H^1(\Omega)} \leq C[\ln(1/h)]^r|\nabla\chi|_{L^2(\Omega)} \quad \forall\chi \in S_0^h, \tag{3.26}$$

where $r = 0$ if $d = 1$ and $r = \frac{1}{2}$ if $d = 2$; see, for example, [15, p. 67]. The quasi-uniformity assumption on T^h for $d = 2$ is not really restrictive in practice, since the real benefits in using non-quasi-uniform meshes only come from those which move in time, tracking the fronts of u and v, for which the present analysis is not appropriate; see [13] for such work on the Stefan problem.

LEMMA 3.4. *Under assumptions* (D6) *we have for all* $\varepsilon \in (0,\varepsilon_0]$ *and* $h > 0$, *provided* $\varepsilon^{-1}kh^2 \leq C$, *and for all* $t \in (0,T]$ *that*

$$\begin{aligned} &\|u_\varepsilon^{h,*} - \hat{u}_\varepsilon^h\|_{E_2(k,t)}^2 + \varepsilon\left|\pi_h[\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|_{L^2(Q_t)}^2 \\ &\quad\leq Ckh^4\left[\varepsilon^{-2}[\ln(1/h)]^{2r}\|u_\varepsilon\|_{L^2(0,t;W^{2,1}(\Omega))}^2 + k\left|\varphi_\varepsilon(g_1^h) - g_2^h\right|_h^2 + \|g_1^h\|_{H^1(\Omega)}^2\right] \\ &\quad\quad + C\left[|P_h^1 g_1 - g_1^h|_{L^2(\Omega)}^2 + \sup_{\chi\in S_0^h}\left\{\frac{|\langle g_2,\chi\rangle - \langle g_2^h,\chi\rangle^h|}{\|\chi\|_{H^1(\Omega)}}\right\}^2\right], \end{aligned} \tag{3.27a}$$

where $r = 0$ *if* $d = 1$ *and* $r = \frac{1}{2}$ *if* $d = 2$ *and*

$$\varepsilon\|v_\varepsilon^{h,*} - \hat{v}_\varepsilon^h\|_{E_1(k,t)}^2 \leq C\left[h^2 + \varepsilon\left|\pi_h[\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|_{L^2(Q_t)}^2 + \varepsilon k^{-1}|g_2 - g_2^h|_{L^2(\Omega)}^2\right]. \tag{3.27b}$$

Proof. Adopting the notation of Lemma 3.3 we have from (3.17a) and (3.22) that

$$\begin{aligned} &\|\hat{e}_\varepsilon^{u,h}\|_{E_2(k,t)}^2 + M^{-1}\varepsilon\left|\pi_h[\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|_{L^2(Q_t)}^2 \\ &\quad\leq \|\hat{e}_\varepsilon^{u,h}\|_{E_2(k,t)}^2 + \int_0^t \left\langle\varphi_\varepsilon(u_\varepsilon(\cdot,s)) - \varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot,s)), \hat{e}_\varepsilon^{u,h}(\cdot,s)\right\rangle^h ds + C\varepsilon^{-1}kh^4 \\ &\quad\equiv T_1 + T_2 + C\varepsilon^{-1}kh^4, \end{aligned} \tag{3.28}$$

where T_1 and T_2 are given by (3.17). We then have from (3.17a), (3.18), (3.26), (3.20), (3.21), (3.3b), and (3.1a) that

$$\begin{aligned} T_1 &\leq C\left[|P_h^1 g_1 - g_1^h|^2_{L^2(\Omega)} + \sup_{\chi\in S_0^h}\left\{\frac{|\langle g_2,\chi\rangle - \langle g_2^h,\chi\rangle^h|}{\|\chi\|_{H^1(\Omega)}}\right\}^2\right] \\ &\quad + \frac{1}{2}\|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)}, \end{aligned} \tag{3.29a}$$

$$\begin{aligned} T_{2,1} &\leq Ch^2\int_0^t \|\pi_h\eta(\cdot\,,s)\|_{H^1(\Omega)}\|\hat{e}_\varepsilon^{u,h}(\cdot\,,s)\|_{H^1(\Omega)}\,ds \\ &\leq Ck^{\frac{1}{2}}h^2\left[\varepsilon^{-\frac{1}{2}} + k^{\frac{1}{2}}\left|\varphi_\varepsilon(g_1^h) - g_2^h\right|_h + \|g_1^h\|_{H^1(\Omega)}\right]\|\hat{e}_\varepsilon^{u,h}\|_{E_2(k,t)}, \end{aligned} \tag{3.29b}$$

$$T_{2,2} \leq Ch^2\|\hat{e}_\varepsilon^{u,h}\|_{E_2(k,t)} + T_3, \tag{3.29c}$$

and

$$\begin{aligned} T_3 &\equiv \left|\int_0^t \left\langle (I-\pi_h)\varphi_\varepsilon(u_\varepsilon(\cdot\,,s)), \hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right\rangle\,ds\right| \\ &\leq C\int_0^t |(I-\pi_h)\varphi_\varepsilon(u_\varepsilon(\cdot\,,s))|_{L^1(\Omega)}\left|\hat{e}_\varepsilon^{u,h}(\cdot\,,s)\right|_{L^\infty(\Omega)}\,ds \\ &\leq Ck^{\frac{1}{2}}[\ln(1/h)]^r\|\hat{e}_\varepsilon^{u,h}\|_{E_2(k,t)}\,|(I-\pi_h)\varphi_\varepsilon(u_\varepsilon)|_{L^2(0,t;L^1(\Omega))}\,. \end{aligned} \tag{3.29d}$$

We have from (3.1a) and (3.25) that

$$\begin{aligned} |(I-\pi_h)\varphi_\varepsilon(u_\varepsilon)|_{L^2(0,t;L^1(\Omega))} &\leq Ch^2|\varphi_\varepsilon(u_\varepsilon)|_{L^2(0,t;W^{2,1}(\Omega))} \\ &\leq C\varepsilon^{-1}h^2\|u_\varepsilon\|_{L^2(0,t;W^{2,1}(\Omega))}. \end{aligned} \tag{3.30}$$

Combining (3.28)–(3.30) yields (3.27a). Finally, (3.27b) follows from (3.23). ∎

THEOREM 3.2. *Let $g_2 \in H^2(\Omega)$ and $g_i^h \equiv \pi_h g_i$, $i = 1,2$. We have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$, provided $\varepsilon^{-1}kh^2 \leq C$ and $t \in (0,T]$:*

(i) *Under assumptions* (D5)

$$\begin{aligned} &|u - \hat{u}_\varepsilon^h|^2_{L^2(Q_t)} + \varepsilon\left|\varphi(u) - \pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|^2_{L^2(Q_t)} + \varepsilon\|v - \hat{v}_\varepsilon^h\|^2_{E_1(k,t)} \\ &\qquad \leq C\left[A_\varepsilon(t)\varepsilon^{(1+p)/(1-p)} + (\varepsilon^{-1}+k)h^2\right]. \end{aligned} \tag{3.31}$$

(ii) *Under assumptions* (D6)

$$|u - \hat{u}_\varepsilon^h|^2_{L^2(Q_t)} \leq C\left[A_\varepsilon(t)\varepsilon^{(1+p)/(1-p)} + \varepsilon^{-2}k^2h^4[\ln(1/h)]^{2r}\right], \tag{3.32a}$$

$$\begin{aligned} &\varepsilon\left|\varphi(u) - \pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|^2_{L^2(Q_t)} + \varepsilon\|v - \hat{v}_\varepsilon^h\|^2_{E_1(k,t)} \\ &\qquad \leq C\left[A_\varepsilon(t)\varepsilon^{(1+p)/(1-p)} + h^2 + \varepsilon^{-2}k^2h^4[\ln(1/h)]^{2r}\right], \end{aligned} \tag{3.32b}$$

where $r = 0$ if $d = 1$ and $r = \frac{1}{2}$ if $d = 2$.

Proof. The result (3.31) follows immediately from (2.7), (3.6a), (3.6d), (3.14), (3.3b), (3.1a), (3.1c), and (3.20b). (3.32) follows similarly with (3.14) replaced by (3.27) and noting (3.10). ∎

COROLLARY 3.2. *Let $g_2 \in H^2(\Omega)$, $g_i^h \equiv \pi_h g_i$, $i = 1,2$, and assumptions* (D5) *hold. Then for all $t \in (0,T]$ we have the following:*

(i) *Under no assumptions on nondegeneracy on choosing* $\varepsilon = Ch^{1-p} \leq \varepsilon_0$, *we have for all* $h \leq h_0(k)$ *that*

$$|(u-\hat{u}_\varepsilon^h)(\cdot\,,t)|_{L^2(\Omega)}+\left|\int_0^t (u-\hat{u}_\varepsilon^h)(\cdot\,,s)\,ds\right|_{H^1(\Omega)}+|\nabla(u-\hat{u}_\varepsilon^h)|_{L^2(Q_T)} \leq C(k)h^{(1+p)/2} \tag{3.33a}$$

and

$$\left|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|_{L^2(Q_T)}+\left|(v-\hat{v}_\varepsilon^h)(\cdot\,,t)\right|_{L^2(\Omega)} \leq C(k)h^p. \tag{3.33b}$$

(ii) *On assuming* (N.D.) *and choosing* $\varepsilon = Ch^{4(1-p)/(5-p)} \leq \varepsilon_0$ *we have for all* $h \leq h_0(k)$ *that*

$$|(u-\hat{u}_\varepsilon^h)(\cdot\,,t)|_{L^2(\Omega)}+\left|\int_0^t (u-\hat{u}_\varepsilon^h)(\cdot\,,s)\,ds\right|_{H^1(\Omega)}+|\nabla(u-\hat{u}_\varepsilon^h)|_{L^2(Q_T)}$$
$$\leq C(k)h^{(3+p)/(5-p)} \tag{3.34a}$$

and

$$\left|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|_{L^2(Q_T)}+\left|(v-\hat{v}_\varepsilon^h)(\cdot\,,t)\right|_{L^2(\Omega)} \leq C(k)h^{(1+3p)/(5-p)}. \tag{3.34b}$$

Proof. The results follow directly from (3.31), (3.14), (3.1a), (3.1c), (3.3b), (2.7), (3.6), and (1.8). ∎

COROLLARY 3.3. *Let* $g_2 \in H^2(\Omega)$, $g_i^h \equiv \pi_h g_i$, $i = 1, 2$, *and assumptions* (D6) *hold. Then for all* $t \in (0, T]$ *we have the following:*

(i) *Under no assumptions on nondegeneracy and on choosing*

$$\varepsilon = C\{h^2[\ln(1/h)]^r\}^{2(1-p)/(3-p)} \leq \varepsilon_0,$$

we have for all $h \leq h_0(k)$,

$$|u-\hat{u}_\varepsilon^h|_{L^2(Q_T)} \leq C(k)\left\{h^2[\ln(1/h)]^r\right\}^{(1+p)/(3-p)}, \tag{3.35a}$$

$$|(u-\hat{u}_\varepsilon^h)(\cdot\,,t)|_{L^2(\Omega)}+\left|\int_0^t (u-\hat{u}_\varepsilon^h)(\cdot\,,s)\,ds\right|_{H^1(\Omega)}+|\nabla(u-\hat{u}_\varepsilon^h)|_{L^2(Q_T)}$$
$$\leq C(k)\min\left(h,\left\{h^2[\ln(1/h)]^r\right\}^{(1+p)/(3-p)}\right), \tag{3.35b}$$

and

$$\left|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|_{L^2(Q_T)}+\left|(v-\hat{v}_\varepsilon^h)(\cdot\,,t)\right|_{L^2(\Omega)} \leq C(k)\left\{h^2[\ln(1/h)]^r\right\}^{2p/(3-p)}. \tag{3.35c}$$

(ii) *On assuming* (N.D.) *and choosing* $\varepsilon = C\left\{h^2[\ln(1/h)]^r\right\}^{4(1-p)/(7-3p)} \leq \varepsilon_0$, *we have for all* $h \leq h_0(k)$,

$$|u-\hat{u}_\varepsilon^h|_{L^2(Q_T)} \leq C(k)\left\{h^2[\ln(1/h)]^r\right\}^{(3+p)/(7-3p)}, \tag{3.36a}$$

$$|(u-\hat{u}_\varepsilon^h)(\cdot\,,t)|_{L^2(\Omega)}+\left|\int_0^t (u-\hat{u}_\varepsilon^h)(\cdot\,,s)\,ds\right|_{H^1(\Omega)}+|\nabla(u-\hat{u}_\varepsilon^h)|_{L^2(Q_T)}$$
$$\leq C(k)\min\left(h,\left\{h^2[\ln(1/h)]^r\right\}^{(3+p)/(7-3p)}\right), \tag{3.36b}$$

and

$$\left|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|_{L^2(Q_T)}+\left|(v-\hat{v}_\varepsilon^h)(\cdot\,,t)\right|_{L^2(\Omega)} \leq C(k)\left\{h^2[\ln(1/h)]^r\right\}^{(1+3p)/(7-3p)}. \tag{3.36c}$$

Proof. The results follow directly from (3.32), (3.27), (3.1a), (3.1c), (3.3b), (3.10), (2.7), (3.6), and (1.8). □

Remark 3.1. Of course the above analysis simplifies for $\{u_\varepsilon^h, v_\varepsilon^h\}$, the unique solution of the less practical scheme $(\mathrm{P}_\varepsilon^h)$. In particular the corresponding version of Lemma 3.3 is as follows: under the assumptions (D3) we have for all $\varepsilon \in (0, \varepsilon_0]$, $h > 0$, and $t \in (0, T]$ that

$$\begin{aligned} &\|u_\varepsilon^{h,*} - u_\varepsilon^h\|_{E_2(k,t)}^2 + \varepsilon|\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(u_\varepsilon^h)|_{L^2(Q_t)}^2 + \varepsilon\|v_\varepsilon^{h,*} - v_\varepsilon^h\|_{E_1(k,t)}^2 \\ &\qquad \leq C\varepsilon^{-1}|u_\varepsilon - u_\varepsilon^{h,*}|_{L^2(Q_t)}^2 \leq C\varepsilon^{-1}kh^4. \end{aligned} \tag{3.37}$$

From this superior rate of convergence over $\{\hat{u}_\varepsilon^h, \hat{v}_\varepsilon^h, \varphi_\varepsilon(\hat{u}_\varepsilon^h)\}$ in (3.14) and (3.27) one can improve, assuming $g_2 \in H^1(\Omega)$, on the error bounds (3.33)–(3.36) above for $\{u_\varepsilon^h, v_\varepsilon^h, \varphi_\varepsilon(u_\varepsilon^h)\}$. For details, see [2].

Remark 3.2. The direct comparison of $\hat{u}_\varepsilon^h$ with $P_h^1 u_\varepsilon$, in place of $u_\varepsilon^{h,*}$, in Lemma 3.4 would require a bound on $\|(I - P_h^1)\partial_s u_\varepsilon\|_{L^2(Q_t)}$. In order not to deteriorate the result (3.27a), this requires a bound on $\|\partial_s u_\varepsilon\|_{L^2(0,t;H^2(\Omega))}$ which in turn requires more regularity on the data g_1, g_2, and f. In addition, the bound (3.37) is degraded by this approach.

Remark 3.3. Let problem $(\hat{\mathrm{P}}^h)$ be the same as $(\hat{\mathrm{P}}_\varepsilon^h)$ with φ_ε replaced by φ. It is a simple matter to prove equivalent versions of Theorem 3.1 and Corollary 3.1 for its unique solution $\{\hat{u}^h, \hat{v}^h\}$. Furthermore it is easy to adapt the result (2.7), see [2], to obtain under assumptions (D4) for all $h > 0$, $\varepsilon \in (0, \varepsilon_0]$, and $t \in (0, T]$ that

$$\begin{aligned} &\|\hat{u}^h - \hat{u}_\varepsilon^h\|_{E_2(k,t)}^2 + k^{-2}\varepsilon\left|\nabla(\hat{u}^h - \hat{u}_\varepsilon^h)(\cdot, t)\right|_{L^2(\Omega)}^2 + \varepsilon\left|\pi_h[\varphi(\hat{u}^h) - \varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|_{L^2(Q_t)}^2 \\ &+\varepsilon\|\hat{v}^h - \hat{v}_\varepsilon^h\|_{E_1(k,t)}^2 \leq C\varepsilon^{(1+p)/(1-p)}. \end{aligned} \tag{3.38}$$

In proving (3.38) we have made no assumptions on the nondegeneracy of $\hat{u}^h$, as such assumptions would be difficult to verify in practice. Therefore, if $g_2 \in H^2(\Omega)$ and $g_i^h \equiv \pi_h g_i$, $i = 1, 2$, then under the assumptions (D5) and (D6) the error bounds (3.33) and (3.35), respectively, hold with $\{\hat{u}_\varepsilon^h, \hat{v}_\varepsilon^h, \varphi_\varepsilon(\hat{u}_\varepsilon^h)\}$ replaced by $\{\hat{u}^h, \hat{v}^h, \varphi(\hat{u}^h)\}$. However, if we know that u satisfies a nondegeneracy condition (N.D.), then from the error estimates (3.31), (3.32), and (3.38) it is better to approximate (P) by $(\hat{\mathrm{P}}_\varepsilon^h)$, with the appropriate choice of ε, rather than $(\hat{\mathrm{P}}^h)$.

One could of course attempt to analyze the error between u and $\hat{u}^h$ without using the regularization procedure by introducing $\{u^{h,*}, v^{h,*}\}$, the unique solution of problem $(\mathrm{P}^{h,*})$, which is the same as $(\mathrm{P}_\varepsilon^{h,*})$ but with φ_ε replaced by φ. If we assume that (1.9c) holds for all a, $b \in \boldsymbol{R}$, as it does for $\varphi(s) \equiv [s]_+^p$, then we have in place of (2.3a) that

$$L^{-1/p}|\varphi(a) - \varphi(b)|^{(1+p)/p} \leq [\varphi(a) - \varphi(b)](a - b) \leq L|a - b|^{1+p}. \tag{3.39}$$

For ease of exposition we consider the error in u^h, where $\{u^h, v^h\}$ is the unique solution of the less practical scheme (P^h). It is a simple matter, using (3.39), to adapt the proof of (3.37) to prove that under the assumptions (D3) for all $h > 0$ and $t \in (0, T]$ that

$$\begin{aligned} &\|u^{h,*} - u^h\|_{E_2(k,t)}^2 + L^{-1/p}\int_0^t |\varphi(u(\cdot, s)) - \varphi(u^h(\cdot, s))|_{L^{(1+p)/p}(\Omega)}^{(1+p)/p} ds \\ &\qquad \leq C\int_0^t |(u - u^{h,*})(\cdot, s)|_{L^{1+p}(\Omega)}^{1+p} ds. \end{aligned} \tag{3.40}$$

Letting $\varepsilon \to 0$ in (3.6a–c) and combining this with the above yields bounds for $u - u^h$ and $\varphi(u) - \varphi(u^h)$. However, bypassing the regularization procedure yields no bound for $v - v^h$ as we have no bound on $v - v^{h,*}$.

4. A fully discrete and practical finite element approximation. In this section we analyze the following fully discrete practical approximation to (P), (the backward Euler time discretization of $(\hat{\mathrm{P}}^h_\varepsilon)$), with timestep $\tau = T/N$:

$(\hat{\mathbf{P}}^{h,\tau}_\varepsilon)$ For $n = 1 \to N$ find $\hat{u}^{h,n}_\varepsilon \in S^h_0$ and $\hat{v}^{h,n}_\varepsilon \in S^h$ such that

$$\tau^{-1}\left\langle(\hat{u}^{h,n}_\varepsilon - \hat{u}^{h,n-1}_\varepsilon) + (\hat{v}^{h,n}_\varepsilon - \hat{v}^{h,n-1}_\varepsilon), \chi\right\rangle^h + \langle\nabla\hat{u}^{h,n}_\varepsilon, \nabla\chi\rangle = \langle f^n, \chi\rangle^h \quad \forall\chi \in S^h_0,$$
$$\tau^{-1}\langle\hat{v}^{h,n}_\varepsilon - \hat{v}^{h,n-1}_\varepsilon, \chi\rangle^h = k\langle\varphi_\varepsilon(\hat{u}^{h,n}_\varepsilon) - \hat{v}^{h,n}_\varepsilon, \chi\rangle^h \quad \forall\chi \in S^h,$$
$$\hat{u}^{h,0}_\varepsilon(\cdot) = g^h_1(\cdot) \qquad \hat{v}^{h,0}_\varepsilon(\cdot) = g^h_2(\cdot),$$

where $f^n(\cdot) \equiv f(\cdot\,, n\tau)$.

Let $\hat{U}_\varepsilon \in L^\infty(0,T;S^h_0)$ and $\hat{V}_\varepsilon \in L^\infty(0,T;S^h)$ be such that for $n = 1 \to N$

$$\hat{U}_\varepsilon(\cdot\,,t) \equiv \hat{u}^{h,n}_\varepsilon(\cdot) \quad \text{and} \quad \hat{V}_\varepsilon(\cdot\,,t) \equiv \hat{v}^{h,n}_\varepsilon(\cdot) \quad \text{if } t \in (\,(n-1)\tau, n\tau].$$

We introduced and analyzed the semidiscrete approximation $(\hat{\mathrm{P}}^h_\varepsilon)$ in the previous section in order to split the error analysis into two (more amenable) parts and in order to isolate the errors due to

(i) spatial discretization by finite elements,

(ii) time discretization.

This is very desirable as one may be interested in alternative time stepping procedures.

THEOREM 4.1. *Let the assumptions* (D4) *hold. Then for all* $\varepsilon \in (0, \varepsilon_0]$, h, $\tau > 0$ *there exists a unique solution* $\{\hat{U}_\varepsilon, \hat{V}_\varepsilon\}$ *to* $(\hat{\mathrm{P}}^{h,\tau}_\varepsilon)$. *Moreover, if* (D5) *holds then*

$$\underline{U} \le \hat{U}_\varepsilon \le \overline{U} \quad \textit{and} \quad \underline{V} \le \hat{V}_\varepsilon \le \overline{V} \qquad \textit{in } Q_T, \tag{4.1}$$

where $\underline{U}$, $\overline{U}$, $\underline{V}$, *and* $\overline{V} \in \boldsymbol{R}$ *are independent of* h, τ, ε, *and* k. *In particular if* g^h_1, g^h_2, *and* $f \ge 0$ *then one can take* $\underline{U} = \underline{V} = 0$.

Proof. This is a discrete analogue of Theorem 2.1; see [2]. ◻

LEMMA 4.1. *Under the assumptions* (D4) *we have for all* $\varepsilon \in (0, \varepsilon_0]$, h, $\tau > 0$, *and* $m = 0 \to N$ *that*

$$\begin{aligned}\|\hat{u}^h_\varepsilon - \hat{U}_\varepsilon\|^2_{E_2(k,m\tau)} &+ \varepsilon|\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon) - \varphi_\varepsilon(\hat{U}_\varepsilon)]|^2_{L^2(Q_{m\tau})} + \varepsilon\|\hat{v}^h_\varepsilon - \hat{V}_\varepsilon\|^2_{E_1(k,m\tau)} \\ &\le C\tau^2\Big\{|\partial_t\hat{u}^h_\varepsilon|^2_{L^2(Q_T)} + (\tau + k^{-1})^{-1}|\nabla(\partial_t\hat{u}^h_\varepsilon)|^2_{L^2(Q_T)} + \big|\partial_t\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]\big|^2_{L^2(Q_T)} \\ &\quad + |\partial_t\hat{v}^h_\varepsilon|^2_{L^2(Q_T)} + |\partial_t[\pi_h f]|^2_{L^2(Q_T)}\Big\}.\end{aligned} \tag{4.2}$$

Proof. Let $E^n_u \equiv (\hat{u}^h_\varepsilon - \hat{U}_\varepsilon)(\cdot\,,n\tau)$, $E^n_v \equiv (\hat{v}^h_\varepsilon - \hat{V}_\varepsilon)(\cdot\,,n\tau)$. Defining $I^n(w)(\cdot) \equiv w(\cdot\,,n\tau) - \tau^{-1}\int_{(n-1)\tau}^{n\tau} w(\cdot\,,s)\,ds$, we then set $\eta^n \equiv I^n(\hat{u}^h_\varepsilon)$, $\xi^n \equiv I^n(\hat{v}^h_\varepsilon)$, $\mu^n \equiv I^n(\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)])$ and $\sigma^n \equiv I^n(f)$. It then follows from $(\hat{\mathrm{P}}^h_\varepsilon)$ and $(\hat{\mathrm{P}}^{h,\tau}_\varepsilon)$ that $E^0_u = E^0_v = 0$ and for $n = 1 \to N$

$$\tau^{-1}\left\langle(E^n_u - E^{n-1}_u) + (E^n_v - E^{n-1}_v), \chi\right\rangle^h + \langle\nabla E^n_u, \nabla\chi\rangle = \langle\nabla\eta^n, \nabla\chi\rangle - \langle\sigma^n, \chi\rangle^h$$
$$\forall\chi \in S^h_0 \tag{4.3a}$$

$$\tau^{-1}\langle E^n_v - E^{n-1}_v, \chi\rangle^h = k\langle[\varphi_\varepsilon(\hat{u}^h_\varepsilon(\cdot\,,n\tau)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot\,,n\tau))] - E^n_v, \chi\rangle^h$$
$$+k\langle\xi^n - \mu^n, \chi\rangle^h \quad \forall\chi \in S^h. \tag{4.3b}$$

We note the following identities, assuming $a^0 = 0$,

$$\sum_{n=1}^m\left[(a^n - a^{n-1})\sum_{i=n}^m b^i\right] \equiv \sum_{n=1}^m a^n b^n, \tag{4.4a}$$

$$\sum_{n=1}^{m}\left[a^n\sum_{i=n}^{m}b^i\right]+\sum_{n=1}^{m}\left[\sum_{i=n}^{m}a^i\right]b^n\equiv\left(\sum_{n=1}^{m}a^n\sum_{n=1}^{m}b^n\right)+\sum_{n=1}^{m}a^nb^n, \tag{4.4b}$$

$$\sum_{n=1}^{m}[(a^n-a^{n-1})a^n]\equiv\frac{1}{2}\left[(a^m)^2+\sum_{n=1}^{m}(a^n-a^{n-1})^2\right]. \tag{4.4c}$$

Choosing $\chi\equiv\sum_{i=n}^{m}E_u^i$ in (4.3a), then summing the equations from $n=1\to m$, and noting (4.4a, 4.4b) yields

$$\begin{aligned}
&\tau\sum_{n=1}^{m}\langle E_u^n+E_v^n,E_u^n\rangle^h+\frac{1}{2}\left[\left|\nabla\left(\tau\sum_{n=1}^{m}E_u^n\right)\right|^2_{L^2(\Omega)}+\tau^2\sum_{n=1}^{m}|\nabla E_u^n|^2_{L^2(\Omega)}\right]\\
&\quad=\left\{\left\langle\nabla\left(\tau\sum_{n=1}^{m}\eta^n\right),\nabla\left(\tau\sum_{n=1}^{m}E_u^n\right)\right\rangle+\tau^2\sum_{n=1}^{m}\langle\nabla\eta^n,\nabla E_u^n\rangle\right.\\
&\qquad\left.-\tau\sum_{n=1}^{m}\left\langle\nabla\left(\tau\sum_{i=n}^{m}\eta^i\right),\nabla E_u^n\right\rangle\right\}\\
&\qquad-\left\{\left\langle\tau\sum_{n=1}^{m}\sigma^n,\tau\sum_{n=1}^{m}E_u^n\right\rangle^h+\tau^2\sum_{n=1}^{m}\langle\sigma^n,E_u^n\rangle^h-\tau\sum_{n=1}^{m}\left\langle\tau\sum_{i=n}^{m}\sigma^i,E_u^n\right\rangle^h\right\}.
\end{aligned} \tag{4.5}$$

Choosing $\chi\equiv E_u^n$ in (4.3a), then summing the equations from $n=1\to m$ and noting (4.4c) yields

$$\begin{aligned}
&\frac{1}{2}\left[|E_u^m|_h^2+\sum_{n=1}^{m}|E_u^n-E_u^{n-1}|_h^2\right]+\sum_{n=1}^{m}\langle E_v^n-E_v^{n-1},E_u^n\rangle^h+\tau\sum_{n=1}^{m}|\nabla E_u^n|^2_{L^2(\Omega)}\\
&\quad=\tau\sum_{n=1}^{m}\left[\langle\nabla\eta^n,\nabla E_u^n\rangle-\langle\sigma^n,E_u^n\rangle^h\right].
\end{aligned} \tag{4.6}$$

From (4.5) and (4.6) it follows that

$$\begin{aligned}
\mathcal{E}\equiv\;&\tau\sum_{n=1}^{m}|E_u^n|_h^2+\frac{1}{2}\left[\left|\nabla\left(\tau\sum_{n=1}^{m}E_u^n\right)\right|^2_{L^2(\Omega)}+\tau^2\sum_{n=1}^{m}|\nabla E_u^n|^2_{L^2(\Omega)}\right]\\
&+k^{-1}\left\{\frac{1}{2}\left[|E_u^m|_h^2+\sum_{n=1}^{m}|E_u^n-E_u^{n-1}|_h^2\right]+\tau\sum_{n=1}^{m}|\nabla E_u^n|^2_{L^2(\Omega)}\right\}\\
&+\tau\sum_{n=1}^{m}\left\langle\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot,n\tau))-\varphi_\varepsilon(\hat{U}_\varepsilon(\cdot,n\tau)),E_u^n\right\rangle^h\\
=\;&-k^{-1}\tau\sum_{n=1}^{m}\left\langle\tau^{-1}(E_v^n-E_v^{n-1})+kE_v^n-k\left[\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot,n\tau))\right.\right.\\
&\left.\left.-\varphi_\varepsilon(\hat{U}_\varepsilon(\cdot,n\tau))\right],E_u^n\right\rangle^h+\left\{\left\langle\nabla\left(\tau\sum_{n=1}^{m}\eta^n\right)\nabla\left(\tau\sum_{n=1}^{m}E_u^n\right)\right\rangle\right.\\
&\left.+\tau^2\sum_{n=1}^{m}\langle\nabla\eta^n,\nabla E_u^n\rangle-\tau\sum_{n=1}^{m}\left\langle\nabla\left(\tau\sum_{i=n}^{m}\eta^i\right),\nabla E_u^n\right\rangle\right\}
\end{aligned}$$

$$-\left\{\left\langle \tau\sum_{n=1}^{m}\sigma^n, \tau\sum_{n=1}^{m}E_u^n\right\rangle^h + \tau^2\sum_{n=1}^{m}\langle\sigma^n, E_u^n\rangle^h - \tau\sum_{n=1}^{m}\left\langle\tau\sum_{i=n}^{m}\sigma^i, E_u^n\right\rangle^h\right\}$$

$$+k^{-1}\tau\sum_{n=1}^{m}\left[\langle\nabla\eta^n, \nabla E_u^n\rangle - \langle\sigma^n, E_u^n\rangle^h\right]. \tag{4.7}$$

From (4.7) and (4.3b) it follows that

$$\begin{aligned}\mathcal{E} \le C\Bigg\{&\left|\nabla\left(\tau\sum_{n=1}^{m}\eta^n\right)\right|^2_{L^2(\Omega)} + \tau(\tau+k^{-1})\sum_{n=1}^{m}|\nabla\eta^n|^2_{L^2(\Omega)} \\ &+[1+(k\tau)^{-1}]^{-1}\sum_{n=1}^{m}\left|\nabla\left(\tau\sum_{i=n}^{m}\eta^i\right)\right|^2_{L^2(\Omega)} + \left|\tau\sum_{n=1}^{m}\sigma^n\right|^2_h \\ &+\tau(\tau^2+k^{-2})\sum_{n=1}^{m}|\sigma^n|_h^2 + \tau\sum_{n=1}^{m}\left|\tau\sum_{i=n}^{m}\sigma^i\right|^2_h + \tau\sum_{n=1}^{m}|\mu^n-\xi^n|_h^2\Bigg\}.\end{aligned} \tag{4.8}$$

Next we note that

$$\left|\tau\sum_{i=n}^{m}I^i(w)(\cdot)\right|^2 \le (m-n)\tau^2\sum_{i=n}^{m}\left|I^i(w)(\cdot)\right|^2 \tag{4.9a}$$

and

$$\begin{aligned}\tau\sum_{i=n}^{m}\left|I^i(w)(\cdot)\right|^2 &\equiv \tau^{-1}\sum_{i=n}^{m}\left|\int_{(i-1)\tau}^{i\tau}[s-(i-1)\tau]w_s(\cdot,s)\,ds\right|^2 \\ &\le C\tau^2\int_{(n-1)\tau}^{m\tau}|w_s(\cdot,s)|^2\,ds.\end{aligned} \tag{4.9b}$$

Therefore it follows from (4.8), (4.9), and (2.3a) that

$$\begin{aligned}&\tau\sum_{n=1}^{m}|E_u^n|_h^2 + \frac{1}{2}\left[\left|\nabla\left(\tau\sum_{n=1}^{m}E_u^n\right)\right|^2_{L^2(\Omega)} + \tau^2\sum_{n=1}^{m}|\nabla E_u^n|^2_{L^2(\Omega)}\right] \\ &\quad+k^{-1}\left\{\frac{1}{2}\left[|E_u^m|_h^2 + \sum_{n=1}^{m}|E_u^n - E_u^{n-1}|_h^2\right] + \tau\sum_{n=1}^{m}|\nabla E_u^n|^2_{L^2(\Omega)}\right\} \\ &\quad+M^{-1}\varepsilon\tau\sum_{n=1}^{m}\left|\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot,n\tau)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot,n\tau))\right|^2_h \\ &\le C\tau^2\Bigg\{\left|\partial_t\left\{\hat{v}_\varepsilon^h - \pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right\}\right|^2_{L^2(Q_T)} + (\tau+k^{-1})^{-1}|\nabla(\partial_t\hat{u}_\varepsilon^h)|^2_{L^2(Q_T)} \\ &\quad+\int_0^T|\partial_t f(\cdot,t)|_h^2\,dt\Bigg\}.\end{aligned} \tag{4.10}$$

Noting bounds like

$$\begin{aligned}&\int_0^{m\tau}|(\hat{u}_\varepsilon^h - \hat{U}_\varepsilon)(\cdot,t)|_h^2\,dt \\ &\quad\le 2\tau\sum_{n=1}^{m}|E_u^n|_h^2 + 2\sum_{n=1}^{m}\int_{(n-1)\tau}^{n\tau}|\hat{u}_\varepsilon^h(\cdot,t) - \hat{u}_\varepsilon^h(\cdot,n\tau)|_h^2\,dt,\end{aligned} \tag{4.11a}$$

$$\sum_{n=1}^{m}\int_{(n-1)\tau}^{n\tau}|\hat{u}_\varepsilon^h(\cdot\,,t)-\hat{u}_\varepsilon^h(\cdot\,,n\tau)|_h^2\,dt \le \sum_{n=1}^{m}\int_{(n-1)\tau}^{n\tau}\left|\int_{n\tau}^{t}\partial_s\hat{u}_\varepsilon^h(\cdot\,,s)\,ds\right|_h^2 dt$$
$$\le \tau^2\int_0^{m\tau}|\partial_t\hat{u}_\varepsilon^h(\cdot\,,t)|_h^2\,dt, \tag{4.11b}$$

(4.9), and (3.3a) it follows that

$$\|\hat{u}_\varepsilon^h-\hat{U}_\varepsilon\|_{E_2(k,m\tau)}^2$$
$$\le C_1\left\{\tau\sum_{n=1}^{m}|E_u^n|_h^2+\frac{1}{2}\left|\nabla\left(\tau\sum_{n=1}^{m}E_u^n\right)\right|_{L^2(\Omega)}^2+k^{-1}\left[\frac{1}{2}|E_u^m|_h^2\right.\right.$$
$$\left.\left.+\tau\sum_{n=1}^{m}|\nabla E_u^n|_{L^2(\Omega)}^2\right]\right\}+C_2\tau^2\left\{|\partial_t\hat{u}_\varepsilon^h|_{L^2(Q_T)}^2+|\nabla(\partial_t\hat{u}_\varepsilon^h)|_{L^2(Q_T)}^2\right\} \tag{4.12a}$$

and

$$\left|\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)-\varphi_\varepsilon(\hat{U}_\varepsilon)]\right|_{L^2(Q_{m\tau})}^2 \le C_3\tau\sum_{n=1}^{m}\left|\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot\,,n\tau))-\varphi_\varepsilon(\hat{U}_\varepsilon(\cdot\,,n\tau))\right|_h^2$$
$$+C_4\tau^2\left|\partial_t\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right|_{L^2(Q_T)}^2. \tag{4.12b}$$

Finally, choosing $\chi\equiv E_v^n$ in (4.3b), then summing the equations from $n=1\to m$ and noting (4.4c), (4.10), and (4.9) yields

$$\frac{1}{2}k^{-1}\left[|E_v^m|_h^2+\sum_{n=1}^{m}|E_v^n-E_v^{n-1}|_h^2\right]+\tau\sum_{n=1}^{m}|E_v^n|_h^2$$
$$=\tau\sum_{n=1}^{m}\left[\left\langle\left[\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot\,,n\tau))-\varphi_\varepsilon(\hat{U}_\varepsilon(\cdot\,,n\tau))\right],E_v^n\right\rangle^h+\langle\xi^n-\mu^n,E_v^n\rangle^h\right]$$
$$\le C\varepsilon^{-1}\tau^2\left\{\left|\partial_t\left\{\hat{v}_\varepsilon^h-\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\right\}\right|_{L^2(Q_T)}^2+|\nabla(\partial_t\hat{u}_\varepsilon^h)|_{L^2(Q_T)}^2+\int_0^T|\partial_t f(\cdot\,,t)|_h^2\,dt\right\}. \tag{4.13}$$

Similarly to (4.12) we have that

$$\|\hat{v}_\varepsilon^h-\hat{V}_\varepsilon\|_{E_1(k,m\tau)}^2\le C_1\left[\tau\sum_{n=1}^{m}|E_v^n|_h^2+\frac{1}{2}k^{-1}|E_v^m|_h^2\right]+C_2\tau^2|\partial_t\hat{v}_\varepsilon^h|_{L^2(Q_T)}^2. \tag{4.14}$$

Combining (4.10), (4.12), (4.13), and (4.14) yields the desired result (4.2). ∎

COROLLARY 4.1. *Under the assumptions* (D4) *we have for all* $\varepsilon\in(0,\varepsilon_0]$, h, $\tau>0$, *and* $m=0\to N$ *that*

$$\|\hat{u}_\varepsilon^h-\hat{U}_\varepsilon\|_{E_2(k,m\tau)}^2+\varepsilon\left|\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)-\varphi_\varepsilon(\hat{U}_\varepsilon)]\right|_{L^2(Q_{m\tau})}^2+\varepsilon\|\hat{v}_\varepsilon^h-\hat{V}_\varepsilon\|_{E_1(k,m\tau)}^2$$
$$\le C(k)\varepsilon^{-1}\tau^2. \tag{4.15}$$

Proof. The result (4.15) follows immediately from (4.2) and the bounds (3.5). ∎

THEOREM 4.2. *Let* $g_2 \in H^2(\Omega)$ *and* $g_i^h \equiv \pi_h g_i$, $i = 1, 2$. *Then we have under the assumptions*

(i) (D5) *on choosing* $\tau \leq Ch$ *that the error bounds* (3.33) *and* (3.34) *hold for the stated choice of* ε *for* $t = m\tau$, $m = 0 \to N$ *with* $\{\hat{u}_\varepsilon^h, \hat{v}_\varepsilon^h\}$ *replaced by* $\{\hat{U}_\varepsilon, \hat{V}_\varepsilon\}$.

(ii) (D6) *on choosing* $\tau \leq C\varepsilon^{-\frac{1}{2}} h^2[\ln(1/h)]^r$ *that the error bounds* (3.35) *and* (3.36) *hold for the stated choice of* ε, *for* $t = m\tau$, $m = 0 \to N$, *with* $\{\hat{u}_\varepsilon^h, \hat{v}_\varepsilon^h\}$ *replaced by* $\{\hat{U}_\varepsilon, \hat{V}_\varepsilon\}$.

Proof. These results follow from balancing the terms (4.15), (3.31), (3.32), (3.14), (3.27), (2.7), (3.6), and (1.8), and noting (3.1a), (3.1c), (3.3b), and (3.10). Therefore in case (i) it follows that we require $\varepsilon^{-1}\tau^2 \leq C\varepsilon^{-1}h^2$ and in case (ii) $\varepsilon^{-1}\tau^2 \leq C\varepsilon^{-2}\{h^2[\ln(1/h)]^r\}^2$. □

5. A numerical experiment. Finally we discuss some numerical experiments. Unfortunately we are not aware of any simple closed form solution with compact support. We consider the following one-dimensional example: $\Omega \equiv (0, 10)$, $T = 1$, $\varphi(s) \equiv 10s^p$, $k = 0.1$, $f \equiv 0$,

$$g_1(x) \equiv \begin{cases} 0 & \text{for } 0 \leq x \leq 2.5, \\ 0.5[(x - 2.5)/1.25]^4 & \text{for } 2.5 \leq x \leq 3.75, \\ 1 - 0.5[(5 - x)/1.25]^4 & \text{for } 3.75 \leq x \leq 5, \end{cases}$$

and $g_1(x) \equiv g_1(10 - x)$ for $5 \leq x \leq 10$, $g_2 \equiv 0$. The solution of (P) is symmetric; i.e., $u(x,t) = u(10-x,t)$, $v(x,t) = v(10-x,t)$. The supports of u and v are identical for $t > 0$, and the two boundary points of the supports, $s_i(t)$ with $s_1(0) = 2.5$ and $s_2(0) = 7.5$, are decreasing and increasing, respectively; see [10]. For $t \in [0, 1]$ we have that $s_i(t) \in \Omega$ so that u and v have restricted regularity. As the initial conditions are not in equilibrium ($\varphi(g_1) \neq g_2$), in addition to diffusion there is a redistribution process between u and v leading initially to an increase of v in a neighborhood of $x = 5$. Eventually u and v will decrease at each point $x \in \Omega$ as the asymptotic limit for $t \to \infty$ is $u = v \equiv 0$.

As a substitute for an exact solution, we take the solution of the fully discrete problem $(\hat{\mathrm{P}}^{h,\tau})$, i.e., without regularization on a uniform mesh with $h = 1/64$ and $\tau = h^2 = 1/4096$, interpolated linearly in time for $t \in ((n-1)\tau, n\tau)$, $n = 1 \to N = 1/\tau$. As the computation of the $L^2(Q_T)$ norm of the error requires storage of this solution on a large number of time levels (see below), a further refinement was not possible.

Although we expect the exact solution to satisfy the nondegeneracy condition (N.D.), no proof is available. Therefore we do not assume that it holds and choose $\tau = 0.1h$, $\varepsilon = 0.1h^{1-p}$ in accordance with (D5) and $\tau = 0.5ph^{4/(3-p)}$, $\varepsilon = 0.5ph^{4(1-p)/(3-p)}$ in accordance with (D6). The constants 0.1 and $0.5p$ were chosen so that τ and ϵ were reasonably small on the coarsest mesh and that in the (D6) case τ on the finest mesh was not less than that used to compute the "exact solution." We consider the cases $p = 0.1$, 0.5, and 0.9 and compute the solutions of the fully discrete problem with regularization on a uniform mesh, i.e., $(\hat{\mathrm{P}}_\varepsilon^{h,\tau})$, for $h = 1/J$, $J = 2, 4, 8, 16$. The resulting systems of nonlinear algebraic equations at each time step are solved by a modified nonlinear SOR method (see [3]) to an accuracy well below the expected discretization error.

The error $\|u - \hat{U}_\varepsilon\|_{L^2(Q_T)}$ is approximated for practical purposes by

$$E_u^h \equiv \left[\frac{1}{N} \sum_{n=1}^{N} |u(\cdot\,, n\tau) - \hat{u}_\varepsilon^{h,n}(\cdot)|_h^2 \right]^{1/2},$$

TABLE 1
$p = 0.1$.

h^{-1}	$E_u^h \times 10^4$		α_u^h		$E_v^h \times 10^4$		α_v^h	
	(D5)	(D6)	(D5) 0.55	(D6) 0.76	(D5)	(D6)	(D5)	(D6)
2	319.69	184.54			788.27	378.35		
4	125.25	45.79	1.35	2.01	434.92	198.53	0.86	0.93
8	57.26	14.52	1.13	1.66	311.42	110.13	0.48	0.85
16	26.75	4.32	1.10	1.75	186.02	55.60	0.74	0.99

TABLE 2
$p = 0.5$.

h^{-1}	$E_u^h \times 10^4$		α_u^h		$E_v^h \times 10^4$		α_v^h	
	(D5)	(D6)	(D5) 0.75	(D6) 1.2	(D5)	(D6)	(D5)	(D6)
2	288.35	396.04			130.88	203.93		
4	115.17	123.46	1.32	1.68	59.38	65.04	1.14	1.65
8	53.82	39.76	1.10	1.63	29.15	21.61	1.03	1.59
16	25.49	12.17	1.08	1.71	14.50	7.09	1.01	1.61

TABLE 3
$p = 0.9$.

h^{-1}	$E_u^h \times 10^4$		α_u^h		$E_v^h \times 10^4$		α_v^h	
	(D5)	(D6)	(D5) 0.95	(D6) 1.81	(D5)	(D6)	(D5)	(D6)
2	287.70	521.64			107.28	244.48		
4	115.95	142.62	1.31	1.87	52.42	67.56	1.03	1.86
8	54.40	38.50	1.09	1.89	26.27	17.79	1.00	1.93
16	25.82	9.53	1.08	2.01	13.02	4.38	1.01	2.02

where $N\tau \leq T = 1 < (N+1)\tau$. It follows from (4.10), (3.5), (3.3a), $\pi_h u \equiv P_h^1 u$ as $d = 1$, a bound similar to (4.11a), (3.1c), and finally (3.10), (2.15), and (3.5) with $\varepsilon = 0$ that

$$\begin{aligned}
[E_u^h]^2 &\leq \frac{2}{N}\sum_{n=1}^{N} |(u - \hat{u}_\varepsilon^h)(\cdot\,, n\tau)|_h^2 + \frac{2}{N}\sum_{n=1}^{N} |\hat{u}_\varepsilon^h(\cdot\,, n\tau) - \hat{u}_\varepsilon^{h,n}(\cdot)|_h^2 \\
&\leq C\tau \sum_{n=1}^{N} |(\pi_h u - \hat{u}_\varepsilon^h)(\cdot\,, n\tau)|_{L^2(\Omega)}^2 + C\varepsilon^{-1}\tau^2 \\
&\leq C|P_h^1 u - \hat{u}_\varepsilon^h|_{L^2(Q_T)}^2 + C\tau^2 |\partial_t (P_h^1 u - \hat{u}_\varepsilon^h)|_{L^2(Q_T)}^2 + C\varepsilon^{-1}\tau^2 \\
&\leq C|u - \hat{u}_\varepsilon^h|_{L^2(Q_T)}^2 + C|(I - P_h^1)u|_{L^2(Q_T)}^2 \\
&\qquad\qquad + C\tau^2 \left[|\nabla(\partial_t u)|_{L^2(Q_T)}^2 + |\partial_t \hat{u}_\varepsilon^h|_{L^2(Q_T)}^2 + \varepsilon^{-1} \right] \\
&\leq C|u - \hat{u}_\varepsilon^h|_{L^2(Q_T)}^2 + Ch^4 + C\varepsilon^{-1}\tau^2. \qquad (5.1)
\end{aligned}$$

Noting Theorem 3.2 and Corollary 4.1, we see that the approximation E_u^h is of sufficient accuracy. An analogous approximation E_v^h is made for the error $\|v - \hat{V}_\varepsilon\|_{L^2(Q_T)}$. It follows from (4.13), (3.5), and (3.3a) that

$$(5.2)\quad [E_v^h]^2 \leq C\tau \sum_{n=1}^{N} |(v - \hat{v}_\varepsilon^h)(\cdot\,, n\tau)|_{L^2(\Omega)}^2 + C\tau \sum_{n=1}^{N} |(I - \pi_h)v(\cdot\,, n\tau)|_{L^2(\Omega)}^2 + C\varepsilon^{-2}\tau^2.$$

Therefore E_v^h may not be of sufficient accuracy due to the second term on the right-hand side of (5.2), which arises from the use of $|\cdot|_h$ instead of $|\cdot|_{L^2(\Omega)}$ in the definition of E_v^h. Nevertheless, for practical convenience we use this approximation.

We estimate the rate of convergence of E_u^h and E_v^h, by setting

$$\alpha_u^h = \frac{\ln[E_u^{2h}/E_u^h]}{\ln 2} \qquad \text{and} \qquad \alpha_v^h = \frac{\ln[E_v^{2h}/E_v^h]}{\ln 2}.$$

Inspecting Tables 1–3 we see that the actual convergence rates for the approximations $\hat{U}_\varepsilon$ and $\hat{V}_\varepsilon$ are better than that predicted by the theory, which appear in the α_u^h column next to the appropriate assumptions; i.e., $\alpha_u^h = (1+p)/2$ and $2(1+p)/(3-p)$ under assumptions (D5) and (D6), respectively.

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