Euler-Poincaré equations for *G*-Strands

To cite this article: Darryl D Holm and Rossen I Ivanov 2014 *J. Phys.: Conf. Ser.* **482** 012018

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Abstract.
The G-strand equations for a map $\mathbb{R}\times\mathbb{R}$ into a Lie group G are associated to a G-invariant Lagrangian. The Lie group manifold is also the configuration space for the Lagrangian. The G-strand itself is the map $g(t,s):\mathbb{R}\times\mathbb{R}\to G$, where t and s are the independent variables of the G-strand equations. The Euler-Poincaré reduction of the variational principle leads to a formulation where the dependent variables of the G-strand equations take values in the corresponding Lie algebra $\mathfrak{g}$ and its co-algebra, $\mathfrak{g}^*$ with respect to the pairing provided by the variational derivatives of the Lagrangian.

We review examples of different G-strand constructions, including matrix Lie groups and diffeomorphism group. In some cases the G-strand equations are completely integrable 1+1 Hamiltonian systems that admit soliton solutions.

1. Introduction

We give a brief account of the G-strand construction, which gives rise to equations for a map $\mathbb{R}\times\mathbb{R}$ into a Lie group G associated to a G-invariant Lagrangian. Our presentation reviews our previous works [7, 5, 6, 3, 8] and is aimed to illustrate the G-strand construction with several simple but instructive examples. The following examples are reviewed here:

(i) $SO(3)$-strand equations for the so-called continuous spin chain. The equations reduce to the integrable chiral model in their simplest (bi-invariant) case.

(ii) $SO(3)$ - anisotropic chiral model, which is also integrable,

(iii) Diff$(\mathbb{R})$-strand equations. These equations are in general non-integrable; however they admit solutions in $2+1$ space-time with singular support (e.g., peakons). Peakon-antipeakon collisions governed by the Diff$(\mathbb{R})$-strand equations can be solved *analytically*, and potentially they can be applied in the theory of image registration.

2. Ingredients of Euler–Poincaré theory for Left G-Invariant Lagrangians

Let G be a Lie group. A map $g(t,s):\mathbb{R}\times\mathbb{R}\to G$ has two types of tangent vectors, $\dot g := g_t\in TG$ and $g' := g_s \in TG$. Assume that the Lagrangian density function $L(g,\dot g,g')$ is left G-invariant. The left G–invariance of L permits us to define $l:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ by

$$L(g,\dot g,g') = L(g^{-1}g, g^{-1}\dot g, g^{-1}g') \equiv l(g^{-1}\dot g, g^{-1}g').$$

Conversely, this relation defines for any reduced lagrangian $l = l(\mathsf{u},\mathsf{v}) : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ a left G-invariant function $L: TG\times TG\to\mathbb{R}$ and a map $g(t,s):\mathbb{R}\times\mathbb{R}\to G$ such that

$$\mathsf{u}(t,s) := g^{-1}g_t(t,s) = g^{-1}\dot g(t,s) \quad\text{and}\quad \mathsf{v}(t,s) := g^{-1}g_s(t,s) = g^{-1}g'(t,s).$$

Lemma 1. *The left-invariant tangent vectors* $\mathsf{u}(t,s)$ *and* $\mathsf{v}(t,s)$ *at the identity of* G *satisfy*

$$\mathsf{v}_t - \mathsf{u}_s = -\,\mathrm{ad}_{\mathsf{u}}\mathsf{v}\,. \qquad (1)$$

Proof. The proof is standard and follows from equality of cross derivatives $g_{ts} = g_{st}$.
Equation (1) is usually called a ***zero-curvature relation***. □

Theorem 2 (Euler-Poincaré theorem for left-invariant Lagrangians)**.**
With the preceding notation, the following two statements are equivalent:

i *Variational principle on $TG \times TG$ $\delta \int_{t_1}^{t_2} L(g(t,s), \dot{g}(t,s), g'(t,s))\, ds\, dt = 0$ holds, for variations $\delta g(t,s)$ of $g(t,s)$ vanishing at the endpoints in t and s. The function $g(t,s)$ satisfies Euler–Lagrange equations for L on G, given by*

$$\frac{\partial L}{\partial g} - \frac{\partial}{\partial t}\frac{\partial L}{\partial g_t} - \frac{\partial}{\partial s}\frac{\partial L}{\partial g_s} = 0.$$

ii *The constrained variational principle*[1]

$$\delta \int_{t_1}^{t_2} l(\mathsf{u}(t,s), \mathsf{v}(t,s))\, ds\, dt = 0$$

holds on $\mathfrak{g} \times \mathfrak{g}$, using variations of $\mathsf{u} := g^{-1}g_t(t,s)$ and $\mathsf{v} := g^{-1}g_s(t,s)$ of the forms

$$\delta\mathsf{u} = \dot{\mathsf{w}} + \mathrm{ad}_{\mathsf{u}}\mathsf{w} \quad \textit{and} \quad \delta\mathsf{v} = \mathsf{w}' + \mathrm{ad}_{\mathsf{v}}\mathsf{w}\,,$$

*where $\mathsf{w}(t,s) := g^{-1}\delta g \in \mathfrak{g}$ vanishes at the endpoints. The **Euler–Poincaré** equations hold on $\mathfrak{g}^* \times \mathfrak{g}^*$ (**G-strand equations**)*

$$\frac{d}{dt}\frac{\delta l}{\delta \mathsf{u}} - \mathrm{ad}^*_{\mathsf{u}}\frac{\delta l}{\delta \mathsf{u}} + \frac{d}{ds}\frac{\delta l}{\delta \mathsf{v}} - \mathrm{ad}^*_{\mathsf{v}}\frac{\delta l}{\delta \mathsf{v}} = 0 \quad \& \quad \partial_s \mathsf{u} - \partial_t \mathsf{v} = [\,\mathsf{u},\, \mathsf{v}\,] = \mathrm{ad}_{\mathsf{u}}\mathsf{v}$$

where $(\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*)$ is defined via $(\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g})$ in the dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ by,

$$\left\langle \mathrm{ad}^*_{\mathsf{u}}\frac{\delta \ell}{\delta \mathsf{u}}\,,\, \mathsf{v} \right\rangle_{\mathfrak{g}} = \left\langle \frac{\delta \ell}{\delta \mathsf{u}}\,,\, \mathrm{ad}_{\mathsf{u}}\mathsf{v} \right\rangle_{\mathfrak{g}}.$$

In 1901 Poincaré in his famous work proves that, when a Lie algebra acts locally transitively on the configuration space of a Lagrangian mechanical system, the well known Euler-Lagrange equations are equivalent to a new system of differential equations defined on the product of the configuration space with the Lie algebra. These equations are called now in his honor Euler-Poincaré equations. In modern language the contents of the Poincaré's article [12] is presented for example in [4, 11]. English translation of the article [12] can be found as Appendix D in [4].

3. G-strand equations on matrix Lie algebras

Denoting $\mathsf{m} := \delta\ell/\delta\mathsf{u}$ and $\mathsf{n} := \delta\ell/\delta\mathsf{v}$ in $\mathfrak{g}^*$, the G-strand equations become

$$\mathsf{m}_t + \mathsf{n}_s - \mathrm{ad}^*_{\mathsf{u}}\mathsf{m} - \mathrm{ad}^*_{\mathsf{v}}\mathsf{n} = 0 \quad \text{and} \quad \partial_t\mathsf{v} - \partial_s\mathsf{u} + \mathrm{ad}_{\mathsf{u}}\mathsf{v} = 0.$$

For G a semisimple *matrix Lie group* and $\mathfrak{g}$ its *matrix Lie algebra* these equations become

$$\begin{aligned} \mathsf{m}_t^T + \mathsf{n}_s^T + \mathrm{ad}_{\mathsf{u}}\mathsf{m}^T + \mathrm{ad}_{\mathsf{v}}\mathsf{n}^T &= 0, \\ \partial_t\mathsf{v} - \partial_s\mathsf{u} + \mathrm{ad}_{\mathsf{u}}\mathsf{v} &= 0 \end{aligned} \tag{2}$$

where the ad-invariant pairing for semisimple matrix Lie algebras is given by

$$\left\langle \mathsf{m}\,,\, \mathsf{n} \right\rangle = \frac{1}{2}\,\mathrm{tr}(\mathsf{m}^T\mathsf{n}),$$

the transpose gives the map between the algebra and its dual $(\,\cdot\,)^T : \mathfrak{g} \to \mathfrak{g}^*$. For semisimple matrix Lie groups, the adjoint operator is the matrix commutator. Examples are studied in [7, 6, 3].

[1] As with the basic Euler–Poincaré equations, this is not strictly a variational principle in the same sense as the standard Hamilton's principle. It is more like the Lagrange d'Alembert principle, because we impose the stated constraints on the variations allowed.

4. Lie-Poisson Hamiltonian formulation

Legendre transformation of the Lagrangian $\ell(\mathsf{u},\mathsf{v}) : \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ yields the Hamiltonian $h(\mathsf{m},\mathsf{v}) : \mathfrak{g}^*\times\mathfrak{g}\to\mathbb{R}$

$$h(\mathsf{m},\mathsf{v}) = \Big\langle \mathsf{m}\,,\,\mathsf{u}\Big\rangle - \ell(\mathsf{u},\mathsf{v})\,. \tag{3}$$

Its partial derivatives imply

$$\frac{\delta l}{\delta \mathsf{u}} = \mathsf{m}\,,\quad \frac{\delta h}{\delta \mathsf{m}} = \mathsf{u} \quad\text{and}\quad \frac{\delta h}{\delta \mathsf{v}} = -\,\frac{\delta \ell}{\delta \mathsf{v}} = \mathsf{v}.$$

These derivatives allow one to rewrite the Euler-Poincaré equation solely in terms of momentum m as

$$\begin{aligned}\partial_t\mathsf{m} &= \mathrm{ad}^*_{\delta h/\delta\mathsf{m}}\,\mathsf{m} + \partial_s\frac{\delta h}{\delta\mathsf{v}} - \mathrm{ad}^*_{\mathsf{v}}\,\frac{\delta h}{\delta\mathsf{v}}\,,\\ \partial_t\mathsf{v} &= \partial_s\frac{\delta h}{\delta\mathsf{m}} - \mathrm{ad}_{\delta h/\delta\mathsf{m}}\,\mathsf{v}\,.\end{aligned} \tag{4}$$

Assembling these equations into Lie-Poisson Hamiltonian form gives,

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathsf{m}\\ \mathsf{v}\end{bmatrix} = \begin{bmatrix}\mathrm{ad}^*(\,\cdot\,)\mathsf{m} & \partial_s - \mathrm{ad}^*_{\mathsf{v}}\\ \partial_s + \mathrm{ad}_{\mathsf{v}} & 0\end{bmatrix}\begin{bmatrix}\delta h/\delta\mathsf{m}\\ \delta h/\delta\mathsf{v}\end{bmatrix} \tag{5}$$

The Hamiltonian matrix in equation (5) also appears in the Lie-Poisson brackets for Yang-Mills plasmas, for spin glasses and for perfect complex fluids, such as liquid crystals.

5. Example: The Euler-Poincaré PDEs for the $SO(3)$-strand and the chiral model. The 2-time spatial and body angular velocities on $\mathfrak{so}(3)$

Let us make the following explicit identification:

$$\mathsf{u} = \begin{pmatrix} 0 & -u_3 & u_2\\ u_3 & 0 & -u_1\\ -u_2 & u_1 & 0\end{pmatrix} \in\mathfrak{g} \quad\leftrightarrow\quad \mathbf{u}\equiv\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}\in\mathbb{R}^3 \tag{6}$$

and similarly for $\mathbf{v}$. In terms of the corresponding group element $g(s,t)$, describing rotation, $\mathsf{u}(t,s) = g^{-1}\partial_t g(t,s)$ and $\mathsf{v}(t,s) = g^{-1}\partial_s g(t,s)$ resemble 2 body angular velocities. For $G = SO(3)$ and Lagrangian $\ell(\mathbf{u},\mathbf{v}) : \mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$, in $1+1$ space-time the Euler-Poincaré equation becomes

$$\frac{\partial}{\partial t}\frac{\delta\ell}{\delta\mathbf{u}} + \mathbf{u}\times\frac{\delta\ell}{\delta\mathbf{u}} = -\left(\frac{\partial}{\partial s}\frac{\delta\ell}{\delta\mathbf{v}} + \mathbf{v}\times\frac{\delta\ell}{\delta\mathbf{v}}\right)\,, \tag{7}$$

and its auxiliary equation becomes

$$\frac{\partial}{\partial t}\mathbf{v} = \frac{\partial}{\partial s}\mathbf{u} + \mathbf{v}\times\mathbf{u}\,. \tag{8}$$

The Hamiltonian form of these equations on $\mathfrak{so}(3)^*$ are obtained from the Legendre transform relations

$$\frac{\delta\ell}{\delta\mathbf{u}} = \mathbf{m}\,,\quad \frac{\delta h}{\delta\mathbf{m}} = \mathbf{u}\quad\text{and}\quad\frac{\delta h}{\delta\mathbf{v}} = -\,\frac{\delta\ell}{\delta\mathbf{v}}\,.$$

Hence, the Euler-Poincaré equation implies the Lie-Poisson Hamiltonian structure in vector form

$$\partial_t\begin{bmatrix}\mathbf{m}\\ \mathbf{v}\end{bmatrix} = \begin{bmatrix}\mathbf{m}\times & \partial_s + \mathbf{v}\times\\ \partial_s + \mathbf{v}\times & 0\end{bmatrix}\begin{bmatrix}\delta h/\delta\mathbf{m}\\ \delta h/\delta\mathbf{v}\end{bmatrix}.$$

This Poisson structure appears in various other theories, such as complex fluids and filament dynamics. When

$$\ell = \frac{1}{2}\int(\mathbf{u}\cdot A\mathbf{u} + \mathbf{v}\cdot B\mathbf{v})\,ds \tag{9}$$

this is the $SO(3)$ *spin-chain model*, which is in general non-integrable- eq. (7) and (8) give:

$$\frac{\partial}{\partial t}A\mathbf{u} + \mathbf{u}\times A\mathbf{u} + \frac{\partial}{\partial s}B\mathbf{v} + \mathbf{v}\times B\mathbf{v} = 0\,, \tag{10}$$

$$\frac{\partial}{\partial t}\mathbf{v} = \frac{\partial}{\partial s}\mathbf{u} + \mathbf{v}\times\mathbf{u}\,. \tag{11}$$

When $A = -B = 1$, this is the $SO(3)$ *chiral model*, which is an integrable Hamiltonian system.

$$\mathbf{u}_t - \mathbf{v}_s = 0\,, \tag{12}$$

$$\mathbf{v}_t - \mathbf{u}_s + \mathbf{u}\times\mathbf{v} = 0\,. \tag{13}$$

6. Integrability

Some of the G-strands models are well known integrable models. They have a *zero-curvature representation* for two operators L and M of the form

$$L_t - M_s + [L, M] = 0, \tag{14}$$

which is the compatibility condition for a pair of linear equations

$$\psi_s = L\psi, \quad \text{and} \quad \psi_t = M\psi.$$

For the SO(3) chiral model for example these operators are

$$\begin{aligned} L &= \frac{1}{4}\left[(1+\lambda)(\mathsf{u}-\mathsf{v}) - \left(1+\frac{1}{\lambda}\right)(\mathsf{u}+\mathsf{v})\right], \\ M &= -\frac{1}{4}\left[(1+\lambda)(\mathsf{u}-\mathsf{v}) + \left(1+\frac{1}{\lambda}\right)(\mathsf{u}+\mathsf{v})\right]. \end{aligned} \tag{15}$$

Another integrable matrix example: $SO(3)$ anisotropic Chiral model [2]

$$\begin{aligned} \partial_t\mathbf{v}(t,s) - \partial_s\mathbf{u}(t,s) + \mathbf{u}\times P\mathbf{v} - \mathbf{v}\times P\mathbf{u} = 0\,, \\ \partial_s\mathbf{v}(t,s) - \partial_t\mathbf{u}(t,s) - \mathbf{v}\times P\mathbf{v} + \mathbf{u}\times P\mathbf{u} = 0\,. \end{aligned} \tag{16}$$

$P = \mathrm{diag}(P_1, P_2, P_3)$ is a constant diagonal matrix. Under the linear change of variables

$$\mathbf{X} = \mathbf{u} - \mathbf{v} \quad \text{and} \quad \mathbf{Y} = -\mathbf{u} - \mathbf{v} \tag{17}$$

equations (16) acquire the form of the following $SO(3)$ anisotropic chiral model,

$$\begin{aligned} \partial_t\mathbf{X}(t,s) + \partial_s\mathbf{X}(t,s) + \mathbf{X}\times P\mathbf{Y} = 0\,, \\ \partial_t\mathbf{Y}(t,s) - \partial_s\mathbf{Y}(t,s) + \mathbf{Y}\times P\mathbf{X} = 0\,. \end{aligned} \tag{18}$$

The system (18) represents two *cross-coupled* equations for $\mathbf{X}$ and $\mathbf{Y}$. These equations preserve the magnitudes $|\mathbf{X}|^2$ and $|\mathbf{Y}|^2$, so they allow the further assumption that the vector fields $(\mathbf{X}, \mathbf{Y})$ take values on the product of unit spheres $\mathbb{S}^2\times\mathbb{S}^2\subset\mathbb{R}^3\times\mathbb{R}^3$. The anisotropic chiral model is an integrable system and its Lax pair in terms of $(\mathbf{u}, \mathbf{v})$ utilizes the following isomorphism between $\mathfrak{so}(3)\oplus\mathfrak{so}(3)$ and $\mathfrak{so}(4)$:

$$A(\mathbf{u},\mathbf{v}) = \begin{pmatrix} 0 & u_3 & -u_2 & v_1 \\ -u_3 & 0 & u_1 & v_2 \\ u_2 & -u_1 & 0 & v_3 \\ -v_1 & -v_2 & -v_3 & 0 \end{pmatrix}. \tag{19}$$

The system (16) can be recovered as a compatibility condition of the operators

$$L = \partial_s - A(\mathbf{v},\mathbf{u})(\lambda\,\mathrm{Id} + J), \tag{20}$$
$$M = \partial_t - A(\mathbf{u},\mathbf{v})(\lambda\,\mathrm{Id} + J), \tag{21}$$

where the diagonal matrix J is defined by

$$J = -\frac{1}{2}\mathrm{diag}(P_1, P_2, P_3, P_1 + P_2 + P_3). \tag{22}$$

This Lax pair is due to Bordag and Yanovski [1]. The $O(3)$ anisotropic chiral model can be derived as an Euler-Poincaré equation from a Lagrangian with quadratic kinetic and potential energy. The details are presented in [7].

Remark 3. *If* $\mathsf{P} = \mathrm{Id}$*, equations (16) recover the* $SO(3)$ *chiral model.*

7. The Diff(ℝ)-strand system

The constructions described briefly in the previous sections can be easily generalized in cases where the Lie group is the group of the Diffeomorphisms. Consider Hamiltonian which is a right-invariant bilinear form given by the H^1 Sobolev inner product

$$H(u,v) \equiv \frac{1}{2}\int_{\mathcal{M}} (uv + u_x v_x)dx. \tag{23}$$

The manifold $\mathcal{M}$ is $\mathbb{S}^1$ or in the case when the class of smooth functions vanishing rapidly at $\pm\infty$ is considered, we will allow $\mathcal{M} \equiv \mathbb{R}$.
Let us introduce the notation $u(g(x)) \equiv u \circ g$. If $g(x) \in G$, where $G \equiv \mathrm{Diff}(\mathcal{M})$, then

$$H(u,v) = H(u \circ g, v \circ g)$$

is a right-invariant H^1 metric.
Let us further consider an one-parametric family of diffeomprphisms, $g(x,t)$ by defining the t - evolution as

$$\dot{g} = u(g(x,t),t), \qquad g(x,0) = x, \qquad \text{i.e.} \qquad \dot{g} = u \circ g \in T_g G; \tag{24}$$

$u = \dot{g} \circ g^{-1} \in \mathfrak{g}$, where $\mathfrak{g}$, the corresponding Lie-algebra is the algebra of vector fields, $\mathrm{Vect}(\mathcal{M})$. Now we recall the following result:

Theorem 4. *(A. Kirillov, 1980, [9, 10]) The dual space of* $\mathfrak{g}$ *is a space of distributions but the subspace of local functionals, called the regular dual* $\mathfrak{g}^*$ *is naturally identified with the space of quadratic differentials* $m(x)dx^2$ *on* $\mathcal{M}$*. The pairing is given for any vector field* $u\partial_x \in Vect(\mathcal{M})$ *by*

$$\langle m dx^2, u\partial_x\rangle = \int_{\mathcal{M}} m(x)u(x)dx$$

The coadjoint action coincides with the action of a diffeomorphism on the quadratic differential:

$$\mathrm{Ad}^*_g : \quad m dx^2 \mapsto m(g) g_x^2 dx^2$$

and

$$\mathrm{ad}^*_u = 2u_x + u\partial_x$$

Indeed, a simple computation shows that

$$\langle \mathrm{ad}^*_{u\partial_x} m dx^2, v\partial_x\rangle = \langle m dx^2, [u\partial_x, v\partial_x]\rangle = \int_{\mathcal{M}} m(u_x v - v_x u)dx =$$
$$\int_{\mathcal{M}} v(2mu_x + um_x)dx = \langle (2mu_x + um_x)dx^2, v\partial_x\rangle,$$

i.e. $\mathrm{ad}^*_u m = 2u_x m + um_x$.

The Diff(ℝ)-strand system arises when we choose $G = \mathrm{Diff}(\mathbb{R})$. For a two-parametric group we have two tangent vectors

$$\partial_t g = u \circ g \quad \text{and} \quad \partial_s g = v \circ g\,,$$

where the symbol $\circ$ denotes composition of functions.
In this right-invariant case, the G-strand PDE system with reduced Lagrangian $\ell(u,v)$ takes the form,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \frac{\partial}{\partial s}\frac{\delta \ell}{\delta v} &= -\operatorname{ad}^*_u \frac{\delta \ell}{\delta u} - \operatorname{ad}^*_v \frac{\delta \ell}{\delta v}\,,\\ \frac{\partial v}{\partial t} - \frac{\partial u}{\partial s} &= \operatorname{ad}_u v\,.\end{aligned} \tag{25}$$

Of course, the distinction between the maps $(u,v) : \mathbb{R}\times\mathbb{R} \to \mathfrak{g}\times\mathfrak{g}$ and their pointwise values $(u(t,s), v(t,s)) \in \mathfrak{g}\times\mathfrak{g}$ is clear. Likewise, for the variational derivatives $\delta\ell/\delta u$ and $\delta\ell/\delta v$.

8. The Diff(ℝ)-strand Hamiltonian structure

Upon setting $m = \delta\ell/\delta u$ and $n = \delta\ell/\delta v$, the right-invariant Diff(ℝ)-strand equations in (25) for maps $\mathbb{R}\times\mathbb{R} \to G = \mathrm{Diff}(\mathbb{R})$ in one spatial dimension may be expressed as a system of two 1+2 PDEs in (t,s,x),

$$\begin{aligned} m_t + n_s &= -\operatorname{ad}^*_u m - \operatorname{ad}^*_v n = -(um)_x - m u_x - (vn)_x - n v_x\,,\\ v_t - u_s &= -\operatorname{ad}_v u = -u v_x + v u_x\,.\end{aligned} \tag{26}$$

The Hamiltonian structure for these Diff(ℝ)-strand equations is obtained by Legendre transforming to

$$h(m,v) = \langle m,\, u\rangle - \ell(u,\, v)\,.$$

One may then write the equations (26) in Lie-Poisson Hamiltonian form as

$$\frac{d}{dt}\begin{bmatrix} m \\ v \end{bmatrix} = \begin{bmatrix} -\operatorname{ad}^*(\,\cdot\,)m & \partial_s + \operatorname{ad}^*_v \\ \partial_s - \operatorname{ad}_v & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta m = u \\ \delta h/\delta v = -\,n \end{bmatrix}. \tag{27}$$

9. Peakon solutions of the Diff(ℝ)-strand equations

With the following choice of Lagrangian,

$$\ell(u,v) = \frac{1}{2}\|u\|^2_{H^1} - \frac{1}{2}\|v\|^2_{H^1}\,, \tag{28}$$

the corresponding Hamiltonian is positive-definite and the Diff(ℝ)-strand equations (26) admit peakon solutions in *both* momenta

$$m = u - u_{xx} \quad \text{and} \quad n = -(v - v_{xx}),$$

with continuous velocities u and v. This is a two-component generalization of the CH equation.

Theorem 5. *The* Diff(ℝ)*-strand equations (26) admit singular solutions expressible as linear superpositions summed over* $a \in \mathbb{Z}$

$$\begin{aligned} m(s,t,x) &= \sum_a M_a(s,t)\delta(x - Q^a(s,t))\,,\\ n(s,t,x) &= \sum_a N_a(s,t)\delta(x - Q^a(s,t))\,,\\ u(s,t,x) &= K * m = \sum_a M_a(s,t)K(x,Q^a)\,,\\ v(s,t,x) &= -K * n = -\sum_a N_a(s,t)K(x,Q^a)\,,\end{aligned} \tag{29}$$

that are peakons *in the case that* $K(x,y) = \frac{1}{2}e^{-|x-y|}$ *is the Green function the inverse Helmholtz operator* $(1-\partial_x^2)$*:*

$$(1-\partial_x^2)K(x,0) = \delta(x)$$

The solution parameters $\{Q^a(s,t), M_a(s,t), N_a(s,t)\}$ with $a \in \mathbb{Z}$ that specify the singular solutions (29) are determined by the following set of evolutionary PDEs in s and t, in which we denote $K^{ab} := K(Q^a, Q^b)$ with integer summation indices $a, b, c, e \in \mathbb{Z}$:

$$\begin{aligned}
\partial_t Q^a(s,t) &= u(Q^a, s, t) = \sum_b M_b(s,t) K^{ab}\,,\\
\partial_s Q^a(s,t) &= v(Q^a, s, t) = -\sum_b N_b(s,t) K^{ab}\,,\\
\partial_t M_a(s,t) &= -\,\partial_s N_a - \sum_c (M_a M_c - N_a N_c)\frac{\partial K^{ac}}{\partial Q^a} \quad \text{(no sum on } a),\\
\partial_t N_a(s,t) &= -\partial_s M_a + \sum_{b,c,e} (N_b M_c - M_b N_c)\frac{\partial K^{ec}}{\partial Q^e}(K^{eb} - K^{cb})(K^{-1})_{ae}\,.
\end{aligned} \tag{30}$$

The last pair of equations in (30) may be solved as a system for the momenta, i.e., Lagrange multipliers (M_a, N_a), then used in the previous pair to update the support set of positions $Q^a(t,s)$.

10. Single-peakon solution of the of the Diff($\mathbb{R}$)-strand system

The single-peakon solution of the Diff($\mathbb{R}$)-strand equations (26) is straightforward to obtain from (30). Combining the equations in (30) for a single peakon shows that $Q^1(s,t)$ satisfies the Laplace equation,

$$(\partial_s^2 - \partial_t^2) Q^1(s,t) = 0\,.$$

Thus, any function $h(s,t)$ that solves the wave equation provides a solution $Q^1 = h(s,t)$. From the first two equations in (30)

$$M_1(s,t) = \frac{1}{K_0} h_t(s,t) \qquad N_1(s,t) = \frac{1}{K_0} h_s(s,t),$$

where $K_0 = K(0,0)$.
The solutions for the single-peakon parameters Q^1, M_1 and N_1 depend only on one function $h(s,t)$, which in turn depends on the (s,t) boundary conditions. The shape of the Green's function comes into the corresponding solutions for the peakon profiles

$$u(s,t,x) = M_1(s,t) K(x, Q^1(s,t))\,, \qquad v(s,t,x) = -N_1(s,t) K(x, Q^1(s,t))\,.$$

11. Peakon-Antipeakon collisions on a Diff($\mathbb{R}$)-strand

Denote the relative spacing $X(s,t) = Q^1 - Q^2$ for the peakons at positions $Q^1(t,s)$ and $Q^2(t,s)$ on the real line and the Green's function $K = K(X)$. Then the first two equations in (30) imply

$$\begin{aligned}
\partial_t X &= (M_1 - M_2)(K_0 - K(X))\,,\\
\partial_s X &= -(N_1 - N_2)(K_0 - K(X))\,,
\end{aligned} \tag{31}$$

where $K_0 = K(0)$.
The second pair of equations in (30) may then be written as

$$\begin{aligned}
\partial_t M_1 &= -\partial_s N_1 - (M_1 M_2 - N_1 N_2) K'(X)\,,\\
\partial_t M_2 &= -\partial_s N_2 + (M_1 M_2 - N_1 N_2) K'(X)\,,\\
\partial_t N_1 &= -\partial_s M_1 + (N_1 M_2 - M_1 N_2)\frac{K_0 - K}{K_0 + K} K'(X)\,,\\
\partial_t N_2 &= -\partial_s M_2 + (N_1 M_2 - M_1 N_2)\frac{K_0 - K}{K_0 + K} K'(X)\,.
\end{aligned} \tag{32}$$

Asymptotically, when the peakons are far apart, the system (32) simplifies, since $\frac{K_0 - K}{K_0 + K} \to 1$ and $K'(X) \to 0$ as $|X| \to \infty$.

The system (32) has two immediate conservation laws obtained from their sums and differences,

$$\begin{aligned}\partial_t(M_1+M_2) &= -\,\partial_s(N_1+N_2)\,,\\ \partial_t(N_1-N_2) &= -\partial_s(M_1-M_2)\,.\end{aligned} \tag{33}$$

These may be resolved by setting

$$\begin{aligned} M_1-M_2 &= \frac{\partial_t X}{K_0-K}\,, \qquad N_1-N_2 = -\frac{\partial_s X}{K_0-K}\,,\\ M_1+M_2 &= \partial_s\phi\,, \qquad N_1+N_2 = -\,\partial_t\phi\,,\end{aligned} \tag{34}$$

and introducing two potential functions, X and ϕ, for which equality of cross derivatives will now produce the system of equations (31) and (32).

12. A simplification.

A simplification arises if $\phi = 0$, in which case the collision is perfectly antisymmetric, as seen from equation (34). This is the peakon-antipeakon collision, for which the equation for X reduces to

$$(\partial_t^2-\partial_s^2)X + \frac{K'}{2(K_0-K)}(X_t^2-X_s^2) = 0\,. \tag{35}$$

This equation can be easily rearranged to produce a linear equation:

$$(\partial_t^2-\partial_s^2)F(X) = 0\,, \quad \text{where} \quad F(X) = \int_{X_0}^{X}(K_0-K(Y))^{-1/2}\,dY\,. \tag{36}$$

When $K(Y) = \frac{1}{2}e^{-|Y|}$, we have

$$F(X) = \sqrt{2}\int_{X_0}^{X}\frac{1}{\sqrt{1-e^{-|Y|}}}\,dY. \tag{37}$$

We can take for simplicity $X_0 = 0$, this would change $F(X)$ only by a constant. The computation gives

$$F(X) = 2\sqrt{2}\,\mathrm{sign}(X)\cosh^{-1}\left(e^{|X|/2}\right)$$

. Hence the solution $X(t,s)$ can be expressed in terms of any solution $h(t,s)$ of the linear wave equation $(\partial_t^2-\partial_s^2)h(t,s) = 0$ as

$$X(t,s) = \pm\ln\left(\cosh^2(h(t,s))\right)\,. \tag{38}$$

$h(t,s)$ is any solution of the wave equation.

$$M_1 = -M_2 = \frac{\partial_t X}{2(K_0-K(X))}\,, \qquad N_1 = -N_2 = -\frac{\partial_s X}{2(K_0-K(X))}.$$

Complex $\mathrm{Diff}(\mathbb{R})$**-strand equations**

The $\mathrm{Diff}(\mathbb{R})$-strands may also be *complexified*. Upon complexifying $(s,t)\in\mathbb{R}^2\to(z,\bar z)\in\mathbb{C}$ where $\bar z$ denotes the complex conjugate of z and setting $\partial_z g = u\circ g$ and $\partial_{\bar z} g = \bar u\circ g$ the Euler-Poincaré G-strand equations in (26) become

$$\begin{aligned}\frac{\partial}{\partial z}\frac{\delta\ell}{\delta u} + \frac{\partial}{\partial\bar z}\frac{\delta\ell}{\delta\bar u} &= -\,\mathrm{ad}^*_u\frac{\delta\ell}{\delta u} - \mathrm{ad}^*_{\bar u}\frac{\delta\ell}{\delta\bar u}\,,\\ \frac{\partial\bar u}{\partial z} - \frac{\partial u}{\partial\bar z} &= \mathrm{ad}_u\bar u\,.\end{aligned} \tag{39}$$

Here the Lagrangian ℓ is taken to be real:

$$\ell(u,\bar u) = \frac{1}{2}\|\nu\|^2_{H^1} = \frac{1}{2}\int u\,(1-\partial_x^2)\,\bar u\,dx. \tag{40}$$

Upon setting $m = \delta\ell/\delta u$, $\bar{m} = \delta\ell/\delta\bar{u}$, for the real Lagrangian ℓ, equations (39) may be rewritten as

$$
\begin{aligned}
m_z + \bar{m}_{\bar{z}} &= -\,\mathrm{ad}^*_u m - \mathrm{ad}^*_{\bar{u}}\,\bar{m} = -(um)_x - mu_x - (\bar{u}\,\bar{m})_x - \bar{m}\,\bar{u}_x\,, \\
\bar{u}_z - u_{\bar{z}} &= -\,\mathrm{ad}_{\bar{u}}\,u = -u\,\bar{u}_x + \bar{u}\,u_x\,,
\end{aligned}
\tag{41}
$$

where the independent coordinate $x \in \mathbb{R}$ is on the real line, although coordinates $(z, \bar{z}) \in \mathbb{C}$ are complex, as are solutions u, and $m = u - u_{xx}$. This is a possible comlexification of the Camassa-Holm equation. These equations are invariant under two involutions, P and C, where

$$P : (x, m) \to (-x, -m) \quad \text{and} \quad C : \text{Complex conjugation.}$$

They admit singular solutions just as before, modulo $\mathbb{R}\times\mathbb{R} \to \mathbb{C}$. For real variables $m = \bar{m}$, $u = \bar{u}$ and real evolution parameter $z = \bar{z} =: t$, they reduce to the CH equation. Their travelling wave solutions and other possible CH complexifications are studied in [5].

Conclusions

The G-strand equations comprise a system of PDEs obtained from the Euler-Poincaré (EP) variational equations for a G-invariant Lagrangian, coupled to an auxiliary *zero-curvature* equation. Once the G-invariant Lagrangian has been specified, the system of G-strand equations in (2) follows automatically in the EP framework. For matrix Lie groups, some of the the G-strand systems are integrable. The single-peakon and the peakon-antipeakon solution of the Diff($\mathbb{R}$)-strand equations (26) depends on a single function of s, t. The *complex* Diff($\mathbb{R}$)-strand equations and their peakon collision solutions have also been solved by elementary means. The stability of the single-peakon solution under perturbations into the full solution space of equations (26) would be an interesting problem for future work.

Acknowledgments

We are grateful for enlightening discussions of this material with F. Gay-Balmaz, T. S. Ratiu and C. Tronci. Work by RII was supported by the Science Foundation Ireland (SFI), under Grant No. 09/RFP/MTH2144. Work by DDH was partially supported by Advanced Grant 267382 FCCA from the European Research Council.

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